2023

MATHEMATICS — HONOURS

Paper : DSE-B-2.3

(Advanced Mechanics)

Full Marks : 65

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

Symbols and Notations have their usual meanings unless otherwise stated.

Group - A

1. Answer the following multiple choice questions with only one correct option. Choose the correct option and justify. 2×10

(a) The number of degrees of freedom of a rigid body rotating about a fixed axis in space is

(i) 1 (ii) 2

(iii) 3 (iv) 6.

(b) A particle of mass m slides under gravity without friction along the parabolic path $y = ax^2$. The Lagrangian of the particle is given by

(i) $L = \frac{1}{2}m\dot{x}^2 - mgax^2$ (ii) $L = \frac{1}{2}m\left(1+4a^2x^2\right)\dot{x}^2 - mgax^2$

(iii) $L = \frac{1}{2}m\dot{x}^2 + mgax^2$ (iv) $L = \frac{1}{2}m\left(1+4a^2x^2\right)\dot{x}^2 + mgax^2$.

(c) Whenever the Lagrangian for a system does not contain a coordinate explicitly,

(i) p_k is a cyclic coordinate (ii) q_k is zero

(iii) p_k is zero (iv) p_k is a constant of motion.

(d) The Lagrangian of a particle of mass m moving in a plane is given by

$$L = \frac{1}{2}m\left(v_x^2 + v_y^2\right) + a\left(xv_y - yv_x\right),$$

where v_x and v_y are velocity components and 'a' is a constant. Then the canonical momentum are given by

(i) $p_x = mv_x$ and $p_y = mv_y$ (ii) $p_x = mv_x + ay$ and $p_y = mv_y + ax$

(iii) $p_x = mv_x - ay$ and $p_y = mv_y + ax$ (iv) $p_x = mv_x - ay$ and $p_y = mv_y - ax$.

(e) If the Lagrangian of a particle moving in one-dimension is given by $L=\frac{\dot{x}^2}{2x}-V(x)$, the Hamiltonian is

(i) $\frac{1}{2}xp^2+V(x)$ (ii) $\frac{1}{2}px^2+V(x)$

(iii) $\frac{1}{2}x^2+V(x)$ (iv) $\frac{1}{2x}p^2+V(x)$.

(f) Which of the following pairs are canonical?

(i) $Q=\frac{1}{p}, P=qp^2$ (ii) $Q=q\cos\beta p, P=q^2\sin\beta p$

(iii) $Q=\log(\sin p), P=q\cot p$ (iv) $P=\frac{1}{2}(p+q), Q=\tan^{-1}\frac{q}{p}$.

(g) The equation of motion of system described by the time-dependent Lagrangian

$L=e^{\gamma t}\left[\frac{1}{2}m\dot{x}^2-V(x)\right]$ is

(i) $m\ddot{x}+\gamma m\dot{x}+\frac{dV}{dx}=0$ (ii) $m\ddot{x}+\gamma m\dot{x}-\frac{dV}{dx}=0$

(iii) $m\ddot{x}-\gamma m\dot{x}+\frac{dV}{dx}=0$ (iv) $m\ddot{x}+\frac{dV}{dx}=0$.

(h) If q_k, p_k are the generalised coordinate and corresponding momentum then the correct relation for Poisson bracket is

(i) $[q_k, q_l]=\delta_{kl}$ (ii) $[q_k, q_k]=1$

(iii) $[q_k, p_k]=0$ (iv) $[q_k, p_k]=1$.

(i) The Routhian for the Lagrangian $L=\dot{r}^2+r^2\dot{\theta}^2+\frac{\mu}{r}$ is given by

(i) $\frac{p_\theta{}^2}{4r^2}+\dot{r}^2+\frac{\mu}{r}$ (ii) $\frac{p_\theta{}^2}{4r^2}-\dot{r}^2+\frac{\mu}{r}$

(iii) $\frac{p_\theta{}^2}{4r^2}-\dot{r}^2-\frac{\mu}{r}$ (iv) $\frac{p_\theta{}^2}{4r^2}+\dot{r}^2-\frac{\mu}{r}$.

(j) A dynamical system having kinetic energy T and potential energy V is described by the Hamiltonian H. Assume that the equations defining the generalised coordinates do not depend on time. Then

(i) H = T + V and is conserved
(ii) H = T + V but it is not conserved
(iii) H ≠ T+V
(iv) L = T + V.

Group - B

Unit - 1

(Marks : 10)

2. Answer *any two* questions :

(a) Define generalised coordinates and a scleronomic system. Find the expression for the kinetic energy of system of N particles and show that for a scleronomous system the kinetic energy is a homogeneous function of generalised velocities. 2+2+1

(b) A bead of mass m slides on a smooth uniform circular wire of radius r which is rotating with a constant angular velocity ω about a fixed vertical diameter. Set up the Lagrangian and find the equation of motion of the bead. 3+2

(c) A particle of mass m moving in a central orbit under inverse square law. Construct the Lagrangian and hence find the equation of motion. 2+3

(d) Define cyclic coordinate. Show that the generalised momentum associated with an cyclic coordinate is a constant of motion for a conservative system. Hence, show that for the motion of a particle in a central force field of potential $V(r)$, angular momentum is conserved. 1+2+2

Unit - 2

(Marks : 15)

3. Answer *any three* questions :

(a) (i) A particle in two dimensions is in a potential $V(x, y) = x + 2y$. Write down the expressions for the canonical momentum p_x and p_y. Hence show that $(p_y - 2p_x)$ is a constant of motion.

(ii) The Hamiltonian of a dynamical system of two degrees of freedom is given by

$$H = q_1 p_1 - q_2 p_2 - aq_1^2 + bq_2^2,$$

where a and b are constants. Show that $F_1 = \frac{p_1 - aq_1}{q_2}$ is a constant of motion. 3+2

(b) A particle with coordinate q and momenta p has the Hamiltonian $H = qp^2 - qp + bp$, where b is a constant. Solve for p and q. 3+2

(c) A particle of mass m is attracted towards a given point by a central force of the form k/r^2, where k is a constant. Assuming that the central force is conservative in nature, write down the expression for the Hamiltonian of the system and derive Hamilton's equations of motion. 3+2

(d) If the Lagrangian $L_0 = \frac{1}{2}m\left(\frac{dq}{dt}\right)^2 - \frac{1}{2}m\omega^2 q^2$ of a dynamical system is modified to $L = L_0 + \alpha q\left(\frac{dq}{dt}\right)$, then show that canonical momentum changes though equation of motion does not change. 2+3

(e) State Hamilton's principle. Derive Hamilton's principle from D'Alembert's principle.

Unit - 3

(Marks : 10)

4. Answer *any two* questions :

(a) A particle of mass m moves on the inside of a frictionless cone having equation $x^2 + y^2 = z^2 \tan^2 \alpha$. Using cylindrical coordinates write the Hamiltonian and Hamiltons equations. 3+2

(b) Show that the integral

$$J_1 = \iint_S \sum_i dq_i \, dp_i$$

taken over an arbitrary two-dimensional surface S of the $2n$-dimensional phase space is invariant under canonical transformation. 5

(c) (i) Using Hamilton's equations of motion, show that the Hamiltonian $H = \frac{p^2}{2m}e^{-rt} + \frac{1}{2}m\omega^2 x^2 e^{rt}$ leads to the equation of motion of a damped harmonic oscillator $\ddot{x} + r\dot{x} + \omega^2 x = 0$.

(ii) If the Lagrangian of a two-dimensional Harmonic oscillator is $L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{1}{2}kr^2$, where k is a constant, then determine the Hamiltonian of the system. 3+2

(d) A system is governed by the Hamiltonian

$$H = \frac{1}{2}(p_x - ay)^2 + \frac{1}{2}(p_y - bx)^2,$$

where a, b are constants and p_x, p_y are momenta conjugate to x and y respectively. Find the values of a and b so that the quantities $(p_x - 3y)$ and $(p_y + 2x)$ are conserved. 5

Unit - 4

(Marks : 10)

5. Answer *any two* questions :

(a) When is a transformation said to be a canonical transformation? A canonical transformation $(q, p) \to (Q, P)$ is made through the generating function $F(q, p) = q^2 p$ on the Hamiltonian $H(q, p) = \dfrac{p^2}{2\alpha q^2} + \dfrac{\beta}{4} q^4$, where α, β are constants. Find the equation of motion for (Q, P). 1+4

(b) Show that the transformation $q = \sqrt{2P}\sin Q$ and $p = \sqrt{2P}\cos Q$ is canonical. Find a generating function for this transformation. 5

(c) If the Poisson bracket of a time-independent dynamical variable u with the Hamiltonian concerned vanishes, show that u is a constant of motion. 5

(d) State and prove Liouville's theorem for a Hamiltonian system. 5

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