2023

MATHEMATICS — HONOURS

Paper : CC-12

(Unit - I + II)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

$\mathbb{Z}$, $\mathbb{R}$ respectively denote the set of all integers and set of all real numbers.

1. Choose the correct answer with proper justification (1 mark for right answer and 1 mark for justification) : 2×10

(a) Let G be a cyclic group of order 36. Then

(i) $G \cong \mathbb{Z}_2 \times \mathbb{Z}_{18}$ (ii) $G \cong \mathbb{Z}_3 \times \mathbb{Z}_{12}$

(iii) $G \cong \mathbb{Z}_4 \times \mathbb{Z}_9$ (iv) $G \cong \mathbb{Z}_6 \times \mathbb{Z}_6$.

(b) Order of the element $(a,(123))$ in $K_4 \times S_3$ is

(i) 2 (ii) 3

(iii) 4 (iv) 6.

(c) It is true that $\text{Aut}(\mathbb{Z}) \cong$

(i) $\mathbb{Z}$ (ii) $\mathbb{Z}_2$

(iii) S_3 (iv) K_4.

(d) If $f : G \to G$ such that $f(a) = a^n$ is an automorphism, then

(i) $a^{n-1} \in Z(G) \forall a \in G$

(ii) $a^{n-1} \notin Z(G)$

(iii) $a^{n-1} \in Z(G)$ if G is abelian

(iv) $a^{n-1} \in Z(G)$ if G is cyclic.

(e) If T be a linear operator on $\mathbb{R}^3$ whose adjoint operator T^* is defined by $T^*(x, y, z) = (2x + z, 0, 0)$, then $T(x, y, z) =$

(i) $(2x, x, 0)$ (ii) $(x, 0, 2x)$

(iii) $(2x, 0, x)$ (iv) $(0, x, 2x)$.

(f) The degree of the minimal polynomial of $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{pmatrix}$ is

(i) 2 (ii) 3

(iii) 4 (iv) 6.

(g) Let ϕ be the linear functional on $\mathbb{R}^2$ such that $\phi(2, 1) = 15$ and $\phi(1, -2) = -10$. Then $\phi(-2, 7) =$

(i) 21 (ii) 28

(iii) 41 (iv) 24.

(h) If $V(F)$ is an inner product space and $\{\theta\}^{\perp}$ is the orthogonal complement of $\{\theta\}$, where θ is the null vector in V, then which of the following is true?

(i) $\{\theta\}^{\perp} = \phi$ (ii) $\{\theta\}^{\perp} = \{\theta\}$

(iii) $\{\theta\}^{\perp} = V$ (iv) $\{\theta\}^{\perp} \subsetneq V$.

(i) The signature of the quadratic form $5x^2 + y^2 + 10z^2 - 4yz - 10zx$ is

(i) 0 (ii) 1

(iii) 2 (iv) 3.

(j) If $x = (a_1, a_2)$, $y = (b_1, b_2)$ be any two vectors in $V_2(F)$, then the length of the vector (3, 4) with respect to the inner product $(x, y) = a_1\bar{b}_1 + (a_1 + a_2)(\bar{b}_1 + \bar{b}_2)$ is

(i) 5 (ii) 58

(iii) $\sqrt{58}$ (iv) None of these.

Unit - I

(Group Theory)

2. Answer *any four* questions :

(a) Let H and K be two finite cyclic groups of order m and n respectively. Prove that $H \times K$ is a cyclic group if and only if $\gcd(m, n) = 1$. 5

(b) Find the number of elements of order 5 in $\mathbb{Z}_{25} \times \mathbb{Z}_5$. 5

(c) (i) Give an example of a group of order 8 which is not abelian. Justify your answer.

(ii) Show that every abelian group of order 35 is cyclic. (1+1)+3

(d) Find all abelian groups of order 1200. 5

(e) (i) If G be a finite cyclic group of order n, then prove that $\text{Aut}(G) \cong U_n$, where U_n is the group of integers under multiplication modulo n.

(ii) Let $f : G \to G$ be an automorphism of G.

Prove that $o(f(a)) = o(a) \forall a \in G$. 3+2

(f) If a group G is an internal direct product of its two subgroups H and K, then prove that

(i) $H \cap K = \{e\}$, e is the identity element in G.

(ii) G is isomorphic to the external direct product of H and K. 2+3

(g) (i) Prove that a non-cyclic group of order 4 is an internal direct product of two cyclic groups each of order two.

(ii) Prove that if Aut(G) is cyclic, then G is abelian. 3+2

Unit - II

(Linear Algebra)

3. Answer ***any five*** questions :

(a) State and prove Cauchy-Schwartz's inequality in an Euclidean space. 1+4

(b) Using Gram-Schimdt orthonormalisation process, find an orthonormal basis corresponding to the basis $\{(1,0,1), (1,1,0), (1,3,4)\}$ in $\mathbb{R}^3(\mathbb{R})$ using standard inner product. 5

(c) Let $V = \mathbb{R}^3$ and $f_1, f_2, f_3 \in V^*$, V^* is the dual space of V such that

$f_1(x, y, z) = x - 2y$, $f_2(x, y, z) = x + y + z$ and $f_3(x, y, z) = y - 3z$.

Prove that $\{f_1, f_2, f_3\}$ is a basis of V^* and find a basis of V for which $\{f_1, f_2, f_3\}$ is the dual basis of V. 2+3

(d) Define eigenspace of a linear operator T on the vector space V. Find the eigen space of $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(x, y) = (x, 0)$. 2+3

(e) Prove that the dual space of an n dimensional vector space is n dimensional. 5

(f) Diagonalise the matrix $\begin{pmatrix} 6 & 4 & -2 \\ 4 & 12 & -4 \\ -2 & -4 & 13 \end{pmatrix}$. 5

(g) Reduce the quadratic form $2x^2 + 5y^2 + 10z^2 + 4xy + 6xz + 12yz$ into its canonical form. 5

(h) Find a Jordan canonical form of $\begin{pmatrix} -1 & 3 & 0 \\ 0 & 2 & 0 \\ 2 & 1 & -1 \end{pmatrix}$ over the field of real numbers. 5