2023

MATHEMATICS — HONOURS

Paper : CC-8

(Riemann Integration and Series of Functions)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

$\mathbb{N}, \mathbb{R}, \mathbb{Q}, \mathbb{Z}$ denote the sets of natural, real, rational numbers and integers respectively.

1. Answer the following multiple choice questions having only one correct option. Choose the correct option and justify your choice. (1+1)×10

(a) Let P, Q and R be three partitions of [0, 1], where $P=\left(0,\frac{1}{4},\frac{1}{2},\frac{3}{4},1\right)$, $Q=\left(0,\frac{1}{8},\frac{1}{4},\frac{1}{2},\frac{7}{8},1\right)$ and $R=\left(0,\frac{1}{8},\frac{1}{4},\frac{3}{8},\frac{1}{2},\frac{3}{4},1\right)$.

Then,

(i) Q is a refinement of R.

(ii) R is a refinement of P and Q both.

(iii) R is a refinement of P and not a refinement of Q.

(iv) R is not a refinement of P and a refinement of Q.

(b) $f(x)=\begin{cases} c & \text{if } 0\le x\le c \\ 2c & \text{if } c<x\le 1 \end{cases}$

If $\int_0^1 f(x)dx=\frac{7}{16}$, the value of '$c$' is

(i) $\frac{1}{2}$ (ii) $\frac{1}{3}$

(iii) $\frac{1}{4}$ (iv) $\frac{1}{5}$.

(c) The number of points of discontinuities of the function $\phi(x) = \int_0^x [\sqrt{t}]\, dt, 0 \le x \le 2023$, ($[r]$ denotes greatest integer $\le r$) is

(i) 1 (ii) 2023

(iii) 0 (iv) 2021.

(d) Let $f: [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = 1, \; x \in [0, 1] \cap \mathbb{Q}$$
$$= -1, \; x \in [0, 1] - \mathbb{Q}$$

Then,

(i) f is $\mathbb{R}$-integrable.

(ii) $|f|$ is $\mathbb{R}$-integrable and f is also $\mathbb{R}$-integrable.

(iii) $|f|$ is not $\mathbb{R}$-integrable and f is $\mathbb{R}$-integrable.

(iv) $|f|$ is $\mathbb{R}$-integrable but f is not $\mathbb{R}$-integrable.

(e) If $I_1 = \int_0^1 x^{\log\left(\frac{1}{100}\right)} dx$ and $I_2 = \int_1^\infty x^{\log\left(\frac{1}{100}\right)} dx$, then

(i) both I_1 and I_2 exist finitely (ii) only I_1 exists finitely

(iii) only I_2 exists finitely (iv) neither I_1 nor I_2 exist finitely.

(f) $\int_0^1 x^{m-1}\left(\log\frac{1}{x}\right)^{n-1} dx \;(m > 0, n > 0)$ is equal to

(i) $\frac{\Gamma(n)}{n^m}$ (ii) $\frac{\Gamma(m)}{m^n}$

(iii) $\frac{\Gamma(n)}{m^n}$ (iv) $\frac{\Gamma(m)}{m^n}$.

(g) The sequence $\left\{\frac{x^n}{n}\right\}_n$ of functions

(i) converges uniformly to 0 in [0, 1]

(ii) converges uniformly to 1 in [0, 1]

(iii) diverges in [–1, 0]

(iv) converges uniformly to 1 in [–1, 1].

(h) The series $1+\frac{1}{1+x^2}+\frac{1}{(1+x^2)^2}+\ldots$ of functions converges

(i) to $1+\frac{1}{x^2}$ everywhere in $[-1, 1]$ (ii) to $1-\frac{1}{x^2}$ in $[-1, 1]$

(iii) to $1-\frac{1}{x^2}$ in $(-1, 1)$ (iv) to $1+\frac{1}{x^2}$ only in $\mathbb{R}\setminus\{0\}$.

(i) The radius of convergence of the power series $\sum_{n=0}^{\infty}\left[\frac{n!}{n^n}(x+2)^n\right]$ is

(i) $(-2, 2)$ (ii) $(-2-e, -2+e)$

(iii) $(-e, e)$ (iv) $(-e-2, -e+2)$.

(j) If $f(x)$ is an odd function defined on $[-\pi, \pi]$, then the Fourier series of $f(x)$ is of the form

(i) $\frac{1}{2}a_0+\sum_{n=1}^{\infty}a_n\cos nx$ (ii) $\frac{1}{2}a_0+\sum_{n=1}^{\infty}b_n\sin nx$

(iii) $\sum_{n=1}^{\infty}a_n\cos nx$ (iv) $\sum_{n=1}^{\infty}b_n\sin nx$.

$[a_0 \neq 0]$

2. Answer *any three* questions :

(a) (i) Define norm of a partition of $[a, b]$.

(ii) Let $f: [a, b] \to \mathbb{R}$ be bounded on $[a, b]$ and P, Q be any two partitions of $[a, b]$. Then prove that $L(P,f) \le U(Q,f)$ and $L(Q,f) \le U(P,f)$. 1+4

(b) Let $f: [a, b] \to \mathbb{R}$, $\phi : [a, b] \to \mathbb{R}$ be both bounded on $[a, b]$ such that $f(x) = \phi(x)$ except for a finite number of points in $[a, b]$. If f be integrable on $[a, b]$ show that ϕ is also integrable on $[a, b]$.

Also prove that $\int_a^b \phi = \int_a^b f$. 2+3

(c) (i) If $f: [a, b] \to \mathbb{R}$ is continuous and $f(x) > 0$, $\forall x \in [a,b]$ and let $F(x)=\int_a^x f(t)dt$, $x \in [a, b]$, show that F is strictly increasing on $[a, b]$.

(ii) Let $f(x)=x|x|, -1 \le x \le 1$. Find a point $c \in [-1, 1]$, such that $\int_{-1}^{1} f(x)dx = 2f(c)$. 3+2

(d) (i) Let $f : [a, b] \to \mathbb{R}$ be integrable on $[a, b]$. If M is the supremum and m is the infimum of f on $[a, b]$ then, prove that

$$m(b-a) \leq \int_a^b f \leq M(b-a).$$

(ii) Evaluate $\int_0^1 \left(2x\sin\frac{1}{x} - \cos\frac{1}{x}\right)dx$ by applying fundamental theorem of Integral Calculus. 3+2

(e) Give an example of a real valued function on [–1, 1] which has a primitive but is not Riemann integrable on [–1, 1].

Give an example of a real valued function on [–1, 1] which is Riemann integrable but does not have a primitive. 5

3. Answer *any two* questions :

(a) Examine for convergence of $\int_0^\infty \frac{x\tan^{-1}x}{\left(1+x^4\right)^{\frac{1}{3}}}dx$. 5

(b) (i) Prove that $B(m,n) = \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}dx \quad (m,n>0)$.

(ii) Show that $\int_{-1}^1 (1+x)^{p-1}(1-x)^{q-1}\,dx = 2^{p+q-1}B(p,q)$. 3+2

(c) Show that the improper integral $\int_0^1 \frac{1}{x}\sin\frac{1}{x}dx$ is convergent. 5

(d) (i) State Dirichlet's test for improper integral $\int_a^\infty f(x)dx \ (a>0)$.

(ii) Show that the improper integral $\int_1^\infty \frac{x}{1+x^2}\sin x\,dx$ is convergent. 2+3

4. Answer *any four* questions :

(a) Prove that the uniform limit function of a sequence of Riemann integrable functions is Riemann integrable on the domain of definition. 5

(b) Prove that the sequence $\{f_n\}_n$, where $f_n(x)=n^2x\left(1-x^2\right)^n, 0\le x\le 1$, is not uniformly convergent on [0, 1] by using $\lim_{n\to\infty}\int_0^1 f_n(x)dx \ne \int_0^1 f(x)dx$, where '$f$' is the limit function of $\{f_n\}_n$. 5

(c) Let $[a, b]$ be a closed and bounded interval and for each $n \in \mathbb{N}$, let f_n be differentiable on $[a, b]$. If each f_n' be continuous on $[a, b]$ and the series of functions $f_1'+f_2'+f_3'+\ldots$ converges uniformly on $[a, b]$ to a function g and the series $f_1+f_2+f_3+\ldots.$ converges to S on $[a, b]$ then show that $S'(x)=g(x)\ \forall x\in[a, b]$. 5

(d) (i) Examine whether

$$\sum_{n=p}^{\infty}\left[4^{-n}\sin\left(3^n\pi x\right)+\frac{\cos\left(n^2x\right)}{p^n}\right]$$

is uniformly convergent on $\mathbb{R}$, where p is a $+ve$ integer ≥ 2.

(ii) Use Weierstrass's M-test to prove that $\sum_{n=1}^{\infty}\frac{2^n x^{2n-1}}{1+x^{2n}}$ converges uniformly for $|x|\le\frac{1}{2}$. 3+2

(e) Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence $R(>0)$. If the series converges at the end point R of the interval of convergence $(-R, R)$, then show that the series is uniformly convergent on the closed interval $[0, R]$. 5

(f) Assuming the power series expansion for $\frac{1}{1+x^2}$ as $\frac{1}{1+x^2}=1-x^2+x^4-x^6+\ldots$, derive the power series of $\tan^{-1}x$ together with its interval of convergence. From this, find the sum of the infinite series $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots$. 3+2

(g) Find the Fourier series for a periodic function $f:\mathbb{R}\to\mathbb{R}$ of period 2π defined by $f(x)=|x|$, $-\pi\le x\le\pi$. Also find the sum of the series at $x=5\pi$. 3+2

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