2023

MATHEMATICS — HONOURS

Paper : CC-5

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

$\mathbb{R}$ denotes the set of real numbers.

Group - A

(Marks : 20)

1. Answer the following multiple choice questions each having only one correct option. Choose the correct option and justify. (1+1)×10

(a) $\lim_{x\to 0} \frac{\sin[x]}{x} =$

(i) 0 (ii) 1

(iii) − 1 (iv) does not exist.

([x] denotes the largest integer not exceeding x)

(b) Let $f(x) = \begin{cases} \frac{1}{\sin x}, & x \in (0,1) \\ 5, & x = 0 \\ 10, & x = 1 \end{cases}$

then f does not have a limit at

(i) 0 (ii) 1

(iii) $\frac{1}{2}$ (iv) $\frac{3}{4}$.

(c) $f(x)=\begin{cases}\dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}, & x\neq 0\\ 0, & x=0\end{cases}$ has discontinuity at $x=0$.

The type of discontinuity is

(i) removable discontinuity (ii) Jump discontinuity

(iii) Oscillatory discontinuity (iv) Infinite discontinuity.

(d) Which statement is true for the function $f(x)=\begin{cases}3+2x; & -\frac{3}{2}<x\leq 0\\ 3-2x; & 0<x<\frac{3}{2}\end{cases}$?

(i) $f(x)$ is continuous at $x = 0$ and also differentiable at $x = 0$

(ii) $f(x)$ is continuous but not differentiable at $x = 0$

(iii) $f(x)$ is differentiable at $x = 0$

(iv) $f(x)$ is nowhere differentiable in $\left(-\frac{3}{2}, \frac{3}{2}\right)$.

(e) If f is a monotonic function defined on an interval I, then identify the monotonic function on I.

(i) $|f|$ (ii) f^2

(iii) f^3 (iv) $f^2 - f$.

(f) Let $f(x)=\dfrac{\sin x}{x}, x\in\left(0, \frac{\pi}{2}\right)$. Then $f(x)$ is a

(i) strictly decreasing function on $\left(0, \frac{\pi}{2}\right)$

(ii) strictly increasing function on $\left(0, \frac{\pi}{2}\right)$

(iii) neither increasing nor decreasing function on $\left(0, \frac{\pi}{2}\right)$

(iv) None of the above.

(g) If $\lim\limits_{x\to 0}\dfrac{a\sin x-\sin 2x}{\tan^3 x}$ is finite, then choose the correct value of 'a' from the following :

(i) $a = 0$ (ii) $a = 2$

(iii) $a = 1$ (iv) $a = -1$.

(h) Which one of the functions does not satisfy the conditions of Rolle's theorem in [– 1, 1]?

(i) $\frac{1}{x^2+4}$ (ii) $\sqrt{x^2+3}$

(iii) x^2 (iv) $x^{2/3}$.

(i) $f(x) = 1 - |x|$ has

(i) maximum value at $x = 0$

(ii) minimum value at $x = 0$

(iii) neither maximum nor minimum at $x = 0$

(iv) None of the above.

(j) Which one is uniformly continuous on the indicated interval?

(i) $f(x) = x^2$, on $[a, b]$, $a \geqslant 0$ (ii) $f(x) = \sin(1/x)$, on $(0, 1)$

(iii) $f(x) = \frac{1}{x}$, on $(0, 1]$ (iv) $f(x) = x^2$, on $[a, \infty)$, $a > 0$.

Group - B

(Marks : 25)

Answer ***any five*** questions.

2. (a) Let $D \subset \mathbb{R}$, f, g, h be functions on D to $\mathbb{R}$. Let $c \in D'$. If $f(x) \leq g(x) \leq h(x)$ for all $x \in D - \{c\}$, and if $\lim_{x \to c} f(x) = \lim_{x \to c} h(x) = l$, then prove that $\lim_{x \to c} g(x) = l$.

(b) Evaluate $\lim_{x \to 0} \frac{\sin x}{x}$ applying the above. 3+2

3. (a) Prove or Disprove :

If $f : [a, b] \to \mathbb{R}$ is continuous at $c \in (a, b)$, then f is continuous on some neighbourhood of 'c'.

(b) Prove or Disprove :

The function $f(x) = x + [x]$, $x \in [0, 2]$ is piecewise continuous in $[0, 2]$. 2+3

4. (a) A real function f is continuous on $[0, 2]$ and $f(0) = f(2)$. Then show that there exists at least a point $c \in [0, 1]$ such that $f(c) = f(c + 1)$.

(b) $f : [0, 1] \to \mathbb{R}$ is continuous on $[0, 1]$ and f assumes only rational values on $[0, 1]$. Then show that f is constant. 2+3

5. (a) Give an example of a function f defined over an interval I such that f has oscillatory discontinuity at a point in I. Justify your answer.

(b) Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that $f(a) < 0, f(b) > 0$ and $A = \{x \in [a, b] : f(x) < 0\}$. If $w = \sup A$, prove that $f(w) = 0$. 2+3

6. (a) Let $f : (a, b) \to \mathbb{R}$ be a monotonically increasing function.

(i) If f is bounded below, show that $\underset{x \to a+}{Lt} f(x)$ exists.

(ii) If f is unbounded below, show that $\underset{x \to a+}{Lt} f(x) = -\infty$.

(b) Prove or disprove : There exists a monotonic function defined on [0, 1] such that the function is discontinuous at every irrational point in [0, 1]. 3+2

7. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$. For any real k, prove that the set $\{x \in \mathbb{R} : f(x) < k\}$ is an open set in $\mathbb{R}$. Hence or otherwise, prove that $\left\{x \in \mathbb{R} : \sin x = \frac{1}{2024}\right\}$ is a closed set in $\mathbb{R}$. 3+2

8. Let $D \subseteq \mathbb{R}$ and $f : D \to \mathbb{R}$ be uniformly continuous on D. If $\{x_n\}$ be a Cauchy sequence in D, then prove that $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{R}$. Is it true when f is continuous on D? 3+2

9. Prove that the necessary and sufficient condition for a continuous function f on an open bounded interval (a, b) to be uniformly continuous on (a, b) is $\lim_{x \to a+} f(x)$ and $\lim_{x \to b-} f(x)$ both exist finitely. 3+2

Group - C

(Marks : 20)

Answer ***any four*** questions.

10. Find the domain of the derived function f', where $f(x)$ is defined for $x > 0$ as follows 5

$$f(x) = \begin{cases} 1 - x^2, & 0 < x \le 1 \\ \log x, & 1 < x \le 2 \\ \log 2 - 1 + \frac{x}{2}, & x > 2 \end{cases}$$

11. (a) If a function $f(x)$ is derivable on a closed interval $[a, b]$ and $f'(a), f'(b)$ are of opposite signs, then prove that there exists at least one point $c \in (a, b)$ such that $f'(c) = 0$.

(b) Prove that there exists $x \in \left(0, \frac{\pi}{2}\right)$ such that $x = \cos x$. 3+2

12. (a) Show that $\left(1-\dfrac{1}{x}\right)^{x} > \left(1-\dfrac{1}{y}\right)^{y}$, if $x > y > 1$.

(b) Let $f(x+y) = f(x)\,f(y)\ \ \forall\, x, y \in \mathbb{R}$. Prove that f is derivable on $\mathbb{R}$ if f is derivable at 1. 2+3

13. (a) Prove or disprove :

If the function $f(x)$ is defined by

$$f(x) = \begin{cases} x; & x < 1 \\ 2x - 1; & x \geq 1 \end{cases}$$

is increasing at 1 but f is not differentiable at 1.

(b) Find 'a' and 'b' such that $\lim\limits_{x \to 0} \dfrac{ae^{x} + be^{-x} + 2\sin x}{\sin x + x\cos x} = 2.$ 2+3

14. State and prove Cauchy's Mean Value theorem and deduce Lagrange's Mean Value theorem from it. 5

15. Expand e^{x} as an infinite series ($x \in \mathbb{R}$). 5

16. Find the greatest value of $x^{m} y^{n}$ ($x, y > 0$) and $x + y = k$ (k is constant); $m, n > 0$. 5