

2023

MATHEMATICS — HONOURS

Paper : CC-3

(Real Analysis)

Full Marks : 65

*The figures in the margin indicate full marks.**Candidates are required to give their answers in their own words as far as practicable.* \mathbb{N} , \mathbb{Q} , \mathbb{R} denote the set of all natural, rational and real numbers respectively.*Notations and symbols have their usual meanings.*

1. Answer all the following multiple choice questions. For each question 1 mark is for choosing the correct option and 1 mark is for justification. (1+1)×10

(a) The derived set of the set $S = \left\{ \frac{n-1}{n+1} \mid n=1,2,\dots \right\} \cup \{2,3\}$ is

- (i) $\{1\}$ (ii) $\{1\} \cup [2,3]$
 (iii) $\{0\}$ (iv) $\{2,3\}$.

(b) The set $\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{n}{n+1} \right)$ is

- (i) open (ii) closed
 (iii) both open and closed (iv) neither open nor closed.

(c) A countable set of irrationals which is dense in \mathbb{R} is

- (i) the set $\mathbb{R} \setminus \mathbb{Q}$ of all irrationals (ii) $\mathbb{Q} \cup \{\sqrt{p} : p \text{ is a prime}\}$
 (iii) $\{\sqrt{p} : p \text{ is a prime}\}$ (iv) $\{\sqrt{2} - r : r \in \mathbb{Q}\}$.

(d) Let $A = [0, 1] \cap \mathbb{Q}$ and $B = \left\{ \frac{1}{n} : n \in \mathbb{N} \right\}$. Then the set $A - B = \{x - y : x \in A, y \in B\}$ is

- (i) empty (ii) finite
 (iii) enumerable (iv) uncountable.

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(e) The sequence $\left\{ \left(\frac{2}{3}\right)^n + \left(\frac{3}{2}\right)^n \right\}$

(i) converges to 0

(ii) converges to 1

(iii) converges to $\frac{13}{6}$

(iv) diverges to $+\infty$.

(f) Let $u_n = \sin \frac{n\pi}{2}, n \geq 1$. Then the subsequence $\{u_{2n-1}\}$

(i) is a convergent subsequence

(ii) diverges to $+\infty$

(iii) is a convergent subsequence and converges to 1

(iv) is oscillatory.

(g) Let $u_n = \cos \frac{n\pi}{2}, v_n = \sin \frac{n\pi}{2}$. Then, $\limsup_{n \rightarrow \infty} (u_n + v_n)$ is equal to

(i) 0

(ii) 1

(iii) 2

(iv) -1.

(h) Which of the following is not a Cauchy sequence?

(i) $\left\{ \frac{(-1)^n}{n} \right\}$

(ii) $\left\{ \left(1 + \frac{1}{n}\right)^n \right\}$

(iii) $\left\{ \frac{1}{n^n} \right\}$

(iv) $\left\{ 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \right\}$.

(i) The series $1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots$ converges for

(i) $p > 1$

(ii) $p < 1$

(iii) $p \leq 1$

(iv) $p \geq 1$.

(j) The infinite series $\frac{\sin \frac{\pi}{2}}{1 \cdot 2} - \frac{\sin \frac{\pi}{2}}{2 \cdot 3} + \dots + (-1)^{n+1} \frac{\sin \frac{\pi}{2}}{n \cdot (n+1)} + \dots$ is

(i) divergent

(ii) oscillatory

(iii) conditionally convergent

(iv) absolutely convergent.

(3)

Z(2nd Sm.)-Mathematics-H/CC-3/CBCS

Unit - 1

Answer *any four* questions.

2. State LUB Axiom of the set of real numbers, \mathbb{R} . Hence deduce the Archimedean property of \mathbb{R} . 1+4
3. (a) Prove or disprove : A countable set cannot have uncountable number of limit points. 3+2
(b) Show that the set $\{x \in \mathbb{R} : \sin x = 0\}$ is countable. 5
4. State and prove Bolzano Weierstrass Theorem on limit points. 5
5. (a) Find all the isolated points of $S = \left\{ \frac{n-1}{n+1} / n=1,2,3,\dots \right\} \cup (2,3)$. 2+3
(b) Prove or Disprove :
 $S = \bigcup_{n=1}^{\infty} I_n$, where $I_n = \left\{ x \in \mathbb{R} : \left(\frac{1}{3} \right)^n \leq x \leq 1 \right\}$; is a closed set. 2+3
6. (a) Give an example of an unbounded countable subset of \mathbb{R} having no limit points. 2+3
(b) Show that the set S is an open set where $S = \{x \in \mathbb{R} : |x-1| + |x-2| < 3\}$. 5
7. Prove or Disprove : Every infinite bounded set of rational numbers has a limit point in \mathbb{Q} . 5
8. (a) Consider the intervals $S = (0, 2]$ and $T = [1, 3)$. Let S° and T° be the set of interior points of S and T respectively. Then find the set of all interior points of $S \setminus T$. 2+3
(b) Prove or disprove : Every uncountable set has a limit point. 2+3

Unit - 2

Answer *any four* questions.

9. (a) Prove that if the sequence $\{x_n\}_n$ converges to l , then the sequence $\{|x_n|\}_n$ converges to $|l|$. Is the converse true? Justify your answer. (2+1)+(1+1)
(b) Give examples of two non-convergent sequence $\{x_n\}_n$ and $\{y_n\}_n$ such that sequences $\{x_n + y_n\}$ and $\{x_n y_n\}$ both converge. (2+1)+(1+1)
10. (a) Prove or disprove : If $\{x_n\}$ and $\{y_n\}$ be two sequences of real numbers such that $\lim_{n \rightarrow \infty} x_n = 0$ and $\{y_n\}$ is a bounded sequence, then $\lim_{n \rightarrow \infty} (x_n y_n) = 0$. 3+2
(b) If $a > 0$, prove that $\left\{ a^{\frac{1}{n}} - 1 \right\}$ is a null sequence. 3+2

Please Turn Over

11. (a) Define Cauchy sequence of real numbers. Using the definition show that $\left\{\frac{n}{n+1}\right\}$ is a Cauchy sequence. (1+2)+2
- (b) Show that every Cauchy sequence is bounded.

12. Let $\{[a_n, b_n]\}$ be a sequence of closed and bounded intervals, such that

- (a) $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for all $n \in \mathbb{N}$ and (b) $\lim_{n \rightarrow \infty} (b_n - a_n) = 0$.

Show that $\bigcap_{n=1}^{\infty} [a_n, b_n]$ contains exactly one element. 5

13. Prove that the sequence $\{x_n\}$ defined by $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2x_n}$ for all $n \geq 1$ is convergent. Find the limit of the sequence $\{x_n\}$. 3+2

14. Prove that every sequence of real numbers has a monotonic subsequence. 5

15. (a) State the Sandwich theorem.

- (b) Prove that $\lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}} \right) = 1$. 2+3

Unit - 3

Answer *any one* question.

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16. (a) Examine the convergence of the series $\sum_{n=1}^{\infty} \frac{\cos(3^n \pi)}{2^n}$.

(b) Test the convergence of the series $\frac{1}{2} + 2 + \frac{1}{2^2} + 2^2 + \frac{1}{2^3} + 2^3 + \dots$ 3+2

17. (a) By comparison test, show that the series $\frac{1}{1 \cdot 2^2} + \frac{1}{2 \cdot 3^2} + \frac{1}{3 \cdot 4^2} + \dots$ is a convergent series.

(b) Show that the series $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$ is absolutely convergent. 2+3