2022

MATHEMATICS — HONOURS

Paper : CC-4

(Group Theory - I)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

1. Answer all the multiple choice questions. Each question carries 2 marks, 1 mark for correct option and 1 mark for justification. (1+1)×10

(a) Let G be a group and $a \in G$. If $0(a) = 17$, then $0(a^8)$ is

(i) 17 (ii) 16

(iii) 8 (iv) 5

(b) Let $(S, \circ)$ be a semigroup. Let e and e' be left and right identities respectively. Then

(i) e may or may not be equal to e'

(ii) $e \neq e'$

(iii) $e = e'$

(iv) e and e' never exist simultaneously.

(c) Consider the group $\mathbb{Z}^2 = \{(a, b) : a, b \in \mathbb{Z}\}$ under component-wise addition. Then which of the following is a subgroup of $\mathbb{Z}^2$?

(i) $\{(a, b) \in \mathbb{Z}^2 \mid ab = 0\}$ (ii) $\{(a, b) \in \mathbb{Z}^2 \mid 3a + 2b = 15\}$

(iii) $\{(a, b) \in \mathbb{Z}^2 \mid 7 \text{ divides } ab\}$ (iv) $\{(a, b) \in \mathbb{Z}^2 \mid 2 \text{ divides } a \text{ and } 3 \text{ divides } b\}$

(d) In S_5, the permutation (1254)(243)(12) is identical with

(i) (3 4 5) (ii) (5 4 3)

(iii) (3 5 4) (iv) (5 3 4)

(e) Let $(\mathbb{Z}, \circ)$ is a group with $x \circ y = x + y + 2$, $x, y \in \mathbb{Z}$; then the inverse of x is

(i) $-(x + 4)$ (ii) $x^2 + 6$

(iii) $-(x - 4)$ (iv) $x + 2$

(f) Which of the following is true?

(i) $\mathbb{Z}_n$ is cyclic if and only if n is prime

(ii) Every proper subgroup of $\mathbb{Z}_n$ is cyclic

(iii) Every proper subgroup of S_4 is cyclic

(iv) If every proper subgroup of a group is cyclic, then the group is cyclic.

(g) Choose the incorrect statement.

(i) Every homomorphic image of a group G is a quotient group $G/_H$ for some choice of normal subgroup H of G

(ii) Any two infinite groups are isomorphic

(iii) $\mathbb{Z}/_{4\mathbb{Z}} \simeq \mathbb{Z}_4$

(iv) Every proper subgroup of S_3 is cyclic.

(h) The number of group homomorphism from the cyclic groups $(\mathbb{Z}_6, +)$ to $(\mathbb{Z}_4, +)$ is

(i) 0 (ii) 1

(iii) 2 (iv) 3.

(i) $f: 4\mathbb{Z} \to \mathbb{Z}_3$ is defined by $f(4n) = [n]$, $n \in \mathbb{Z}$, then $ker\, f$ is

(i) $3\mathbb{Z}$ (ii) $6\mathbb{Z}$

(iii) $12\mathbb{Z}$ (iv) $\mathbb{Z}$.

(j) Consider the group $(\mathbb{Q}^*, \cdot)$, the multiplicative group of all non-zero rational numbers and its subgroup $\mathbb{Q}^+$, set of all positive rational numbers. Then $[\mathbb{Q}^* : \mathbb{Q}^+]$ is

(i) 2 (ii) 3

(iii) 6 (iv) 8.

Unit - I

2. Answer *any two* questions :

(a) Correct or justify : The set $G = \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} : a \in \mathbb{R}, a \neq 0 \right\}$ forms a group under matrix multiplication and the group is abelian. 5

(b) (i) Let $GL(2, \mathbb{R})$ be the group of all non-singular 2×2 matrices over $\mathbb{R}$. Show that $H = \left\{ \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \in GL(2, \mathbb{R}) : ad \neq 0 \right\}$ is a subgroup of $GL(2, \mathbb{R})$.

(ii) Let $(G, \circ)$ be a group and a, b be two elements of the group. Assume that $0(a) = 5$ and $a^3 \circ b = b \circ a^3$. Then prove that $ab = ba$. 3+2

(c) Establish a necessary and sufficient condition for a nonempty subset of a group to be a subgroup of it. 5

(d) (i) Let $(G, \circ)$ be a group. Suppose that $a, b \in G$ such that $a \circ b = b \circ a$ and $o(a), o(b)$ are relatively prime. Then prove that $o(a \circ b) = o(a) \circ o(b)$.

(ii) Prove that a group G can not be written as the union of two proper subgroups. 3+2

Unit - II

3. Answer *any four* questions :

(a) (i) Let G be a group and $a \in G$ be a unique element in G of order 2. Prove that $ax = xa$ for all $x \in G$.

(ii) Find the order of the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 1 & 5 & 2 & 6 \end{pmatrix} \in S_6$. 3+2

(b) (i) Prove that every group of prime order is cyclic.

(ii) Prove that $(\mathbb{Q}, +)$ is a non-cyclic group. 3+2

(c) (i) Show that S_4 has no elements or order ≥ 5.

(ii) In S_6, let $\rho = (123)$ and $\sigma = (456)$. Find a permutation x in S_6 such that $x \rho x^{-1} = \sigma$. 3+2

(d) (i) Find all distinct left cosets of the subgroup $H = \{e, (123), (132)\}$ in the group S_3.

(ii) How many generators are there in a group of order 23? 3+2

(e) (i) Let $\beta = (123)(145)$. Write β^{99} in cycle form.

(ii) Let α and β belong to S_n. Prove that $\beta \alpha \beta^{-1}$ and α are both even or both odd permutation. 2+3

(f) (i) Let G be an abelian group. Show that the set of all elements of finite order in G forms a subgroup of G.

(ii) Prove that every group of order 4 is commutative. 3+2

(g) (i) Let A and B be subgroups of a group G. If $|A| = p$, a prime number, show that either $A \cap B = \{e\}$ or $A \subseteq B$.

(ii) Consider the group $\mathbb{R}^2$ under component-wise addition of real numbers. Let $H = \{(x, 3x) : x \in \mathbb{R}\}$. Show that H is a subgroup of $\mathbb{R}^2$ and any straight line parallel to $y = 3x$ is a coset of H. 2+3

Unit - III

4. Answer *any three* questions :

(a) (i) Let H be a normal subgroup of G and S be the set of all distinct cosets at H in G. Then prove that $(S, \bullet)$, where '$\bullet$' is defined by $aH \bullet bH = abH$, for all $a, b \in G$ forms a group.

(ii) Let G be a group and H be a subgroup of G such that $[G : H] = 2$. Prove that $x^2 \in H$ if $x \in G$. 3+2

Please Turn Over

(b) Let G be a group of order n. Prove that G is isomorphic to a subgroup of the symmetric group S_n. 5

(c) (i) Let $(G, \bullet)$ be a group in which $(a \bullet b)^3 = a^3 \bullet b^3$ for all $a, b \in G$. Prove that $H = \{x^3 : x \in G\}$ is a normal subgroup of G.

(ii) For a fixed element a in a group $(G, \bullet)$, define $f_a : G \to G$ such that $f_a(x) = a^{-1}.x.a$, for all $x \in G$. Show that f_a is a group isomorphism. 3+2

(d) (i) Prove that any two finite cyclic groups of same order are isomorphic.

(ii) Consider $\mathbb{C}^*$ as the group of non-zero complex number under multiplication of complex number and define $f : \mathbb{C}^* \to \mathbb{C}^*$ by $f(z) = z^6$. Prove that f is a homomorphism. 3+2

(e) (i) Prove that $8\mathbb{Z}/56\mathbb{Z} \simeq \mathbb{Z}_7$.

(ii) State Third Isomorphism theorem in group theory. 3+2
