2021

MATHEMATICS — HONOURS

Paper : CC-9

(Partial Differential Equation and Multivariate Calculus-II)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

All symbols have their usual meaning.

Group – A

(Marks : 20)

1. Answer all questions with proper explanation/justification (one mark for correct answer and one mark for justification) : (1+1)×10

(a) Show that $u(x, y) = \exp\left(-\frac{x}{b}\right) f(ax - by)$, where a, b are arbitrary constants and f is an arbitrary function, satisfies

(i) $bu_x + au_y + u = 0$ (ii) $bu_x - au_y - u = 0$

(iii) $bu_y + au_x + u = 0$ (iv) none of these.

(b) If $u_x = v_y$ and $v_x = -u_y$, then u and v satisfy one of the following relations :

(i) $\nabla^2 u \neq 0, \nabla^2 v \neq 0$ (ii) $\nabla^2 u = 0, \nabla^2 v \neq 0$

(iii) $\nabla^2 u \neq 0, \nabla^2 v = 0$ (iv) $\nabla^2 u = 0, \nabla^2 v = 0$.

(c) Nature of the partial differential equation $u_{xx} - \sqrt{y} u_{xy} + x u_{yy} = \cos\left(x^2 - 2y\right), y \geq 0$ is

(i) hyperbolic if $y > 4x$; parabolic if $y = 4x$; elliptic if $y < 4x$

(ii) elliptic if $y > 4x$; parabolic if $y = 4x$; hyperbolic if $y < 4x$

(iii) hyperbolic if $y = 4x$; parabolic if $y < 4x$; elliptic if $y > 4x$

(iv) none of these.

(d) Characteristic curves of the partial differential equation $u_{xx} + x^2 u_{yy} = 0$, for $x \neq 0$ is given by

(i) $2y - ix^2 = c_1, 2y + ix^2 = c_2$ (ii) $4y - ix^2 = c_1, 4y + ix^2 = c_2$

(iii) $3y - ix^2 = c_1, 3y + ix^2 = c_2$ (iv) none of these.

(e) Which one of the following is a nonlinear partial differential equation?

(i) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$ (ii) $x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} = nz$

(iii) $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 1$ (iv) none of these.

(f) Value of the integral $\iint \cos(x+y)dxdy$ over the domain $x = 0$, $y = \pi$, $y = x$ is

(i) 2 (ii) – 2

(iii) 1 (iv) none of these.

(g) Value of the integral $\int \frac{ds}{x-y}$ along the line $2y = x - 4$ between the points (0, – 2) and (4, 0) is

(i) $\sqrt{5}\log 2$ (ii) $\frac{\sqrt{5}}{2}\log 2$

(iii) $2\sqrt{5}\log 2$ (iv) none of these.

(h) Value of the integral $\iiint xyz\, dxdydz$ over R : [0, 1; 0, 1; 0, 1] is

(i) $\frac{1}{2}$ (ii) $\frac{1}{4}$

(iii) $\frac{1}{8}$ (iv) none of these.

(i) If the order of integration in $\int_0^1 dy \int_0^{\sqrt{y}} f(x, y)dx$ is reversed, then it will take the form

(i) $\int_0^1 dx \int_{x^2}^1 f(x, y)dy$ (ii) $\int_0^1 dx \int_x^1 f(x, y)dy$

(iii) $\int_0^1 dx \int_{\sqrt{x}}^1 f(x, y)dy$ (iv) none of these.

(j) A particle, acted on by constant forces $\left(5\vec{i}+2\vec{j}+\vec{k}\right)$ and $\left(2\vec{i}-\vec{j}-3\vec{k}\right)$, is displaced from the origin to the point $\left(4\vec{i}+\vec{j}-3\vec{k}\right)$. The total work done by the forces is

(i) 33 units (ii) 35 units

(iii) 37 units (iv) none of these.

Group – B

(Marks : 21)

Answer ***any three*** questions.

2. (a) Form the partial differential equation by eliminating the arbitrary function f from the equation $z = x\,f\left(\frac{y}{x}\right)$.

(b) Find the solution of the equation $(y-u)u_x + (u-x)u_y = x-y$ with the Cauchy data $u = 0$ on $xy = 1$.

2+5

3. (a) Apply Charpit's method to find the complete integral of the partial differential equation $px + qy = pq$.

(b) Solve the partial differential equation $u_x^2 + u_y^2 = u$ using $u\,(x, y) = f(x) + g(y)$. 4+3

4. Reduce the second order partial differential equation $yu_{xx} + (x+y)u_{xy} + xu_{yy} = 0$ to a canonical form and hence solve it. 7

5. A stretched string of finite length L is fixed at its ends and is subjected to an initial displacement $u(x, 0) = u_0 \sin\left(\frac{\pi x}{L}\right)$. The string is released from this position with zero initial velocity. Find the resultant motion of the string. 7

6. Solve the following initial-boundary value problem :

$$u_t = 4u_{xx},\ 0 < x < 1,\ t > 0$$
$$u\left(x, 0\right) = x^2(1-x),\ 0 \le x \le 1$$
$$u\left(0, t\right) = 0,\ u(1, t) = 0,\ t \ge 0$$

by variable separation method. 7

Group – C

(Marks : 24)

Answer ***any four*** questions.

7. Using differentiation under the sign of integration, prove that

$$\int_0^{\pi/2} \log\left(a\cos^2\theta + b\sin^2\theta\right)d\theta = \pi\log\left[\frac{1}{2}\left(\sqrt{a}+\sqrt{b}\right)\right],\ a, b > 0.$$ 6

8. By changing the order of integration, prove that $\int_0^1 dx \int_x^{\frac{1}{x}} \frac{y\,dy}{(1+xy)^2\left(1+y^2\right)} = \frac{(\pi-1)}{4}$. 6

9. Evaluate the integral $\iint_E \sqrt{4a^2 - x^2 - y^2}\,dxdy$, where E is the region bounded by the circle

$$x^2 + y^2 = 2ax.$$ 6

10. Find the value of the integral $\iiint_V \frac{dxdydz}{(x+y+z+1)^4}$, where V is the volume enclosed within the tetrahedron formed by the planes $x + y + z = 1$ and $x = 0$, $y = 0$, $z = 0$. 6

11. (a) If $\frac{1}{2}\oint_C (xdy - ydx)$ represents the area bounded by the closed curve C, then find the area bounded by the curve $x = a\cos^3\theta$, $y = a\sin^3\theta$.

(b) Show that the vector field given by $(y + \sin z)\vec{i} + x\vec{j} + (x\cos z)\vec{k}$ is conservative. Find a scalar potential of this field. 3+3

12. (a) Use Stoke's theorem to find the line integral $\int_C \left(x^2y^3dx + dy + zdz\right)$, where C is the circle

$$x^2 + y^2 = a^2,\ z = 0.$$

(b) Apply Divergence theorem to evaluate $\iint_S \vec{F}.\vec{n}\,dS$ where $\vec{F} = \left(x^2 - yz\right)\vec{i} + \left(y^2 - zx\right)\vec{j} + \left(z^2 - xy\right)\vec{k}$ and S is the surface of the rectangular parallelopiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$. Here $\vec{n}$ is the unit outward drawn normal to the surface S. 3+3

13. Show that the volume common to the cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$ is $\frac{16a^3}{3}$. 6
