2021

MATHEMATICS — HONOURS

Paper : CC-8

(Riemann Integration and Series of Functions)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

$\mathbb{N}$, $\mathbb{R}$ denote the sets of natural numbers and real numbers respectively.

1. Answer all the following multiple choice questions having only one correct option. Choose the correct option and justify : (1+1)×10

(a) Let $f:[a,b]\to\mathbb{R}$ be a bounded function and P, Q are partitions of $[a, b]$ such that P is a refinement of Q. Then,

(i) $L(P, f)\leq L(Q, f)$ (ii) $L(P, f)\leq U(Q, f)$

(iii) $U(P, f)\leq L(Q, f)$ (iv) $U(P, f)\geq U(Q, f)$.

(b) Let $f:[0,3]\to\mathbb{R}$ be defined by $f(x) = [x]$, where $[x]$ denotes the greatest integer not exceeding x. Then,

(i) f is not Riemann integrable on [0, 3].

(ii) f is Riemann integrable on [0, 3] and $\int_0^3 f = 0$.

(iii) f is Riemann integrable on [0, 3] and $\int_0^3 f = 2$.

(iv) f is Riemann integrable on [0, 3] and $\int_0^3 f = 3$.

(c) Identify the incorrect statement :

(i) Any subset of a negligible set is negligible.

(ii) Any enumerable set of real numbers is negligible.

(iii) Countable union of negligible sets is negligible.

(iv) If the set of points of discontinuity of a real-valued function is negligible, then the function is monotonic.

(d) Let $f:[0,4]\to\mathbb{R}$ be defined by $f(x) = x^4 - 4x^3 + 10$ and $P = \{1, 2, 3, 4\}$. Then,

(i) $U(P, f) = -40$ (ii) $L(P, f) = 11$

(iii) $U(P, f) = 40$ (iv) $L(P, f) = -40$.

(e) $\int_0^\infty \sqrt{t}\, e^{-t^3} dt$ is equal to

(i) $\frac{\sqrt{\pi}}{3}$ (ii) $\frac{\sqrt{\pi}}{2}$

(iii) $\frac{\sqrt{\pi}}{4}$ (iv) $2\sqrt{\pi}$.

(f) The improper integral $\int_1^\infty \frac{dx}{x^{\mu-2}}$ is convergent if and only if

(i) $\mu = 1$ (ii) $\mu < 2$

(iii) $\mu \geq 2$ (iv) $\mu > 3$.

(g) The radius of convergence of the power series $x + \frac{x^2}{2^2} + \frac{x^3}{3^3} + \frac{x^4}{4^4} + ...$ is

(i) e (ii) $\frac{1}{e}$

(iii) ∞ (iv) 0.

(h) The limit function of $\left\{\frac{x^n}{1+x^n}\right\}_n$ on [0, 2] is

(i) monotonically decreasing (ii) monotonically increasing

(iii) continuous (iv) not monotonic.

(i) Given that the interval of uniform convergence of a power series is (– 4, 2), for suitable a_n, which could be power series?

(i) $\sum_{n=0}^{\infty} a_n (X+3)^n$ (ii) $\sum_{n=0}^{\infty} a_n (X-3)^n$

(iii) $\sum_{n=0}^{\infty} a_n (X+1)^n$ (iv) $\sum_{n=0}^{\infty} a_n (X-1)^n$.

(j) The sum of the Fourier series for the function $f:[-\pi, \pi]\to \mathbb{R}$ defined by

$$f(x)=\begin{cases}-1, -\pi\le x<0\\ -2, 0\le x\le \pi\end{cases} \text{ at } x=\pi \text{ is}$$

(i) $-\frac{1}{2}$ (ii) -2

(iii) $-\frac{3}{2}$ (iv) $\frac{3}{2}$.

2. Answer ***any three*** questions :

(a) State and prove a necessary and sufficient condition for Riemann integrability of a bounded function f defined on $[a, b]$. 1+4

(b) If a real-valued function f is Riemann integrable on $[a, b]$ then prove that $|f|$ is also Riemann integrable on $[a, b]$ and $\left|\int_a^b f\right| \le \int_a^b |f|$. 3+2

(c) (i) If $f:[a,b]\to\mathbb{R}$ is a continuous function, such that $f(x)\ge 0$ on $[a, b]$ and $\int_a^b f=0$, then prove that f is identically zero on $[a, b]$.

(ii) Prove, with justification, $\frac{\pi^2}{9}\le \int_{\pi/6}^{\pi/2} \frac{x}{\sin x}dx \le \frac{2\pi^2}{9}$. 2+3

(d) Let $f(t)=\lim_{n\to\infty}\frac{t^n+1}{t^n+3}, 0\le t\le 2$ and $F(x)=\int_0^x f(t)\,dt, x\ge 0$. Prove that F is continuous at '1' but is not derivable there. 2+3

(e) (i) Prove or disprove : If $f:[a,b]\to\mathbb{R}$ has a primitive on $[a, b]$, then the set of points of discontinuity of f in $[a, b]$ is a negligible set.

(ii) Identify the set of points of discontinuity of the function $f:[0,1]\to\mathbb{R}$ defined by

$$f(x)=\begin{cases}\frac{1}{3^n}, & \text{when } \frac{1}{3^{n+1}}<x\le\frac{1}{3^n} (n=0,1,2,.....)\\ 0, & \text{when } x=0\end{cases}$$

Hence, tell whether f is Riemann integrable on [0, 1]. 2+(2+1)

3. Answer ***any two*** questions :

(a) Let the functions f, g be positive-valued, bounded and Riemann integrable over $[a, X]$ for every $X > a$ such that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty$. If $\int_a^\infty f$ is convergent prove that $\int_a^\infty g$ is also convergent. Is the converse true? Justify your answer. 3+2

(b) Show that $\int_0^\infty \frac{\sin x}{x} dx$ is convergent but $\int_0^\infty \frac{|\sin x|}{x} dx$ is not convergent. 2+3

(c) Prove that $B(m, n) = \int_0^1 \frac{t^{m-1} + t^{n-1}}{(1+t)^{m+n}} dt$ where $m > 0$, $n > 0$. 5

(d) (i) Examine the convergence of $\int_1^2 \frac{\log x}{\sqrt{2-x}} dx$.

(ii) Examine the absolute convergence of $\int_0^\infty \frac{\cos x dx}{\sqrt{1+x^3}}$. 2+3

4. Answer ***any four*** questions :

(a) Let $\{f_n\}_n$ be a sequence of Riemann integrable functions defined on $[a, b]$ and $\{f_n\}_n$ have a uniform limit f on $[a, b]$. Prove that f is Riemann integrable over $[a, b]$.

Moreover, show that $\lim_{n\to\infty} \int_a^b f_n(x) dx = \int_a^b \left(\lim_{n\to\infty} f_n(x) \right) dx$. 3+2

(b) Show that the sequence $\left\{ \frac{nx}{1+n^2x^2} \right\}_n$ of continuous functions defined on $[0, 1]$ has a continuous limit function, although the convergence is not uniform. 2+3

(c) $\sum_n M_n$ is a convergent infinite series of positive real numbers such that $|f_n(x)| \leq M_n$ for all $x \in S$ and for every $n \in \mathbb{N}$. Prove that $\sum_n f_n$ is uniformly convergent on S.

Hence, prove that $\sum_{n=1}^{\infty} \frac{\cos^3 nx}{4n^2+1}$ is uniformly convergent on $[0, \infty)$. 3+2

(d) Examine term-by-term differentiability of $\sum_{n=1}^{\infty} \frac{1}{n^3 + n^4 x^2}$ on $\mathbb{R}$. 5

(e) Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series and $\mu = \overline{\lim} |a_n|^{\frac{1}{n}}$. If $0 < \mu < \infty$, prove that the series is absolutely convergent for $|x| < \frac{1}{\mu}$ and is not convergent for $|x| > \frac{1}{\mu}$. 5

(f) Assuming the power series for $(1+x)^{-1}$ *as* $(1+x)^{-1} = 1 - x + x^2 - x^3 + ...(-1 < x < 1)$, obtain the power series expansion of $\log_e(1+x)$ and find the region of convergence of the power series of $\log_e(1+x)$. 3+2

(g) Find the Fourier series of the function $f(x) = \begin{cases} -1, & -\pi \leq x \leq 0 \\ 1, & 0 < x \leq \pi \end{cases}$.

Also find the sum of the series at $x = 0$ and $x = \frac{\pi}{2}$. 3+2
