2021

MATHEMATICS — HONOURS

Sixth Paper

(Module : XII)

Full Marks : 50

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Symbols have their usual meanings.

Group - A

[Hydrostatics]

(Marks : 25)

Answer ***any two*** questions taking ***one*** from **each section**.

Section - I

1. (a) An elliptic lamina is completely immersed vertically in a liquid with its minor axis horizontal and its centre is at a depth h below the effective surface. Determine the position of the centre of pressure.

(b) A hollow cone, vertex upwards, is three-quarters full of water and is set rotating about its axis which is vertical, with an angular velocity $\sqrt{\frac{8g}{3h}}\cot\alpha$, where α is the semi-vertical angle and h is the height of the cone. Show that the ratio of the thrust on the base to the weight of the water in the vessel is 10 : 3. 7+8

2. (a) A hemispherical bowl is filled with water and two vertical planes are drawn through its central radius, cutting off a semi-lune of the surface. If 2α be the angle between the planes, prove that the angle which the resultant pressure on the surface makes with the vertical is $\tan^{-1}\left(\frac{\sin\alpha}{\alpha}\right)$.

(b) A circular cylinder of radius R, whose axis is vertical is filled with fluid to a depth h with a homogeneous liquid of density ρ. A piston of weight $\pi\rho g\alpha R^2$ which works in the cylinder without friction, is placed on the top of the fluid. Show that if the cylinder and fluid rotate about its axis with gradually increasing angular velocity ω, the piston will begin to rise, when $\omega R = 2\sqrt{g\alpha}$. 8+7

Section - II

3. (a) Prove that the surface of separation of two liquids of different densities, which do not mix, at rest under gravity is a horizontal plane.

(b) A vertical circular cylinder of height $2h$ and radius r, closed at the top, is just filled by equal volumes of two liquids of densities ρ and σ. Show that if the axis be gradually inclined to the vertical, the pressure at the lowest point of the base will never exceed $g(\rho+\sigma)\left(r^2+h^2\right)^{\frac{1}{2}}$. 5+5

4. (a) Show that when a uniform hemisphere of density ρ and radius a floats with its plane base immersed in a homogeneous liquid of density σ, the equilibrium is stable and the metacentric height is $\frac{3}{8}a\left(\frac{\sigma}{\rho}-1\right)$.

(b) If, near the Earth's surface, gravity be assumed to be constant and the temperature in the atmosphere to be given by $t=t_0\left(t-\frac{z}{nH}\right)$ where H is the height of the homogeneous atmosphere, show that the pressure in the atmosphere will be given by the equation $p=p_0\left(1-\frac{z}{nH}\right)^n$. 5+5

Group - B

[Rigid Dynamics]

(Marks : 25)

Answer ***any two*** questions, taking ***one*** from **each section**.

Section - I

5. (a) State D'Alembert's principle. Deduce the general equations of motion from D'Alembert's principle.

(b) A heavy uniform rigid straight rod OA of length $2a$ is free to turn about its end O. If the rod revolves uniformly with angular velocity Ω about the vertical through O making a constant angle $\pi/4$ with the vertical, show that $\Omega^2=\frac{3g}{2\sqrt{2}a}$. (2+5)+8

6. (a) A uniform rod of length $2a$ is placed with one end in contact with a smooth horizontal table and is then allowed to fall; if α be its initial inclination to the vertical, show that its angular velocity when it is inclined at an angle θ is $\left\{\frac{6g}{a}\frac{\cos\alpha-\cos\theta}{1+3\sin^2\theta}\right\}^{1/2}$.

(b) A plank of mass m is initially at rest along a straight line of greatest slope of a smooth plane inclined at an angle α to the horizontal and a man of mass m', starting from the upper end walks down the plank so that it does not move. Show that he will reach the other end in time,

$$\sqrt{\frac{2m'a}{(m+m')g\sin\alpha}}$$ 8+7

Section - II

7. (a) A rough uniform rod of length $2a$ is placed on a rough table at right angles to its edge; if its centre of gravity be initially at a distance b beyond the edge, show that the rod will begin to slide when it has turned through an angle $\tan^{-1}\frac{\mu a^2}{a^2+9b^2}$, where μ is the coefficient of friction.

(b) Two equal uniform rods AB and BC, are freely joined at B and turn about a smooth joint at A. When the rods are in a straight line, ω being the angular velocity of AB and u be the linear velocity of the centre of gravity of BC; BC impinges on a fixed inelastic obstacle at a point O; show that the rods are instantly brought to rest, if $BO = 2a\cdot\frac{2u-a\omega}{3u+2a\omega}$, where $2a$ is the length of the either rod.

5+5

8. (a) A uniform disc of radius a is rolling without slipping along a smooth horizontal table with velocity V, when the highest point becomes suddenly fixed. Prove that the disc will make a complete revolution round the point if $v^2 > 24\,ag$ where g is the acceleration due to gravity.

(b) A circular ring of mass M and radius a lies on a smooth horizontal plane and an insect of mass m resting on it starts walking round it with uniform velocity v relative to the ring. Show that centre of the ring describes a circle with angular velocity $\frac{mv}{(M+2m)a}$. 5+5