2019

MATHEMATICS — HONOURS

Paper : CC–4

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

1. Choose the correct alternative. Justify your answer. Each question carries 2 marks – 1 mark for right answer and 1 mark for justification. 2×10

(a) The order of the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 3 & 5 & 1 & 2 \end{pmatrix}$ is

(i) 4

(ii) 6

(iii) 8

(iv) 12

(b) Which of the following does not form a group under matrix multiplication?

(i) $\left\{\begin{pmatrix} a & b \\ -b & a \end{pmatrix} : a, b \in \mathbb{R}, a^2 + b^2 = 1\right\}$

(ii) $\left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b \in \mathbb{R}, ad - bc = 1\right\}$

(iii) $\left\{\begin{pmatrix} a & b \\ 2b & a \end{pmatrix} : a, b \in \mathbb{Q}, \text{ and } (a, b) \neq (0, 0)\right\}$

(iv) $\left\{\begin{pmatrix} a & b \\ 2b & a \end{pmatrix} : a, b \in \mathbb{R}, \text{ and } (a, b) \neq (0, 0)\right\}$

(c) Number of generators of the additive group Z_{36} is equal to

(i) 6 (ii) 12 (iii) 18 (iv) 36

(d) Let $(G, 0)$ be a group and $a, b \in G$. If $0(a) = 4$ and $a\ 0\ b\ 0\ a^{-1} = b^3$, then for $b \neq e$, the $0(b)$ divides

(i) 3 (ii) 26 (iii) 80 (iv) 9

(e) If $S = \{1, -1, i, -i\}$, then $(S, 0)$ is a cyclic group generated by

(i) $1, -1$

(ii) $1, i$

(iii) $-1, i$

(iv) $i, -i$

(f) Let $(G, 0)$ be a group. A mapping $f : G \to G$ is defined by $f(x) = x^{-1}$, $x \in G$. Then f is

(i) one to one but not onto

(ii) onto but not one to one

(iii) one to one and onto

(iv) none of the above.

(g) Number of group homomorphism from the cyclic group Z_4 to the cyclic group Z_7 is

(i) 7 (ii) 3 (iii) 2 (iv) 1

(h) The centre $\mathbb{Z}(G)$ of a group G is

(i) cyclic group of G

(ii) non-cyclic sub-group of G

(iii) normal sub-group of G

(iv) not normal sub-group of G

(i) Choose the incorrect statement.

(i) Every homomorphic image of a group G is a quotient group G/H for some choice of normal subgroup H of G.

(ii) Any two infinite group are isomorphic.

(iii) $Z/4Z \simeq Z_4$

(iv) None of the above.

(j) The mapping $f : (\mathbb{Z}, +) \to (2\mathbb{Z}, +)$ defined by $f(x) = 2x$ for all $x \in \mathbb{Z}$ is

(i) an injective homomorphism but not surjective

(ii) a surjective homomorphism but not injective

(iii) an isomorphism

(iv) neither injective nor surjective.

Unit – I

2. Answer ***any three*** questions : 5×3

(a) If $(G, 0)$ be a semi group and for $a, b \in G$ the equations $a \, 0 \, x = b$ and $y \, 0 \, a = b$ has a solution in G, then prove that $(G, 0)$ be a group. 5

(b) (i) Let G be a group. Can you express G as union of two proper nontrivial subgroups H and K? Justify your answer.

(ii) Prove that the set of all rational numbers of the form $3^m 6^n$ where m, n are integers is a group under multiplication. 3+2

(c) Show that $C_G(Z(G)) = G$ and $N_G(Z(G)) = G$. 5

(d) (i) In a group $(G, 0)$ if $a^3b^3 = (ab)^3$ and $a^5b^5 = (ab)^5$ for all $a, b \in G$; then prove that G is commutative.

(ii) Let $(G, 0)$ be a finite group of even order. Prove that G contains an element of order 2. 3+2

(e) (i) Let $a, b \in \mathbb{R}$, consider a mapping $f_{a,b} : \mathbb{R} \to \mathbb{R}$ defined by $f_{a,b}(x) = ax + b$ for all $x \in \mathbb{R}$. Let $G = \{f_{a,b} : a, b \in \mathbb{R} \text{ with } a \neq 0\}$. Prove that G forms a group with respect to usual composition of two mappings.

(ii) Let G be a group such that every proper subgroup of G is commutative. Does it necessarily imply G is commutative? Justify your answer. 3+2

Unit – II

3. Answer *any two* questions : 5×2

(a) (i) If p be a prime and a be an integer such that p is not a divisor of a, then show that $a^{p-1} \equiv 1 \pmod p$ by using result in group theory.

(ii) Let G be a group and $a \in G$. Prove that order of a divides order of G. 3+2

(b) (i) Prove that every group of prime order is cyclic.

(ii) Justify the statement — "Every abelian group may not be cyclic". 3+2

(c) (i) Prove that a noncommutative group of order 10 must have a subgroup of order 5.

(ii) Prove that $(\mathbb{Q}, +)$ is a non-cyclic group. 3+2

(d) (i) Let H be a subgroup of a group G and $a, b \in G$. Prove that the left cosets aH and bH are identical if $a^{-1}b \in H$.

(ii) Does there exist any subgroup of order 11 of the symmetric group S_7? Justify your answer. 3+2

Unit – III

4. Answer *any four* questions : 5×4

(a) (i) Let H be a subgroup of a group $(G, 0)$. Prove that for all $g \in G$, $gHg^{-1} = H$ if and only if $gH = Hg$.

(ii) Let H be a subgroup of a group G such that the product of any two left cosets of H is a left coset of H. Prove that H is normal in G. 3+2

(b) State and prove First Isomorphism Theorem. 5

(c) (i) Let $\varphi : G \to G'$ be a group homomorphism. Prove that Im φ is a subgroup of G'.

(ii) Show that the groups $(Q, +)$ and $(Q^+, \cdot)$ are not isomorphic where Q is the set of rational numbers, '+' and '·' are usual addition and multiplication operation. 3+2

(d) If G is an infinite cyclic group, then prove that G has exactly two generators and G is isomorphic to the additive group of integers. 5

(e) Prove that every finite group of order n is isomorphic to a subgroup of S_n. 5

(f) (i) Show that if every cyclic subgroup of a group G is normal then every subgroup of G is normal.

(ii) If H be a subgroup of a commutative group G then the quotient group G/H is commutative. Is the converse true? Justify. 2+3

(g) (i) Let G be a group of order 30 and A, B be two normal subgroups of G of order 2 and 5 respectively. Show that ${G}/{AB}$ contains 3 elements.

(ii) Show that any two finite cyclic groups of same order are isomorphic. 2+3
