2019

MATHEMATICS — HONOURS

Paper : CC–3

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

$\mathbb{R}$, $\mathbb{Q}$, $\mathbb{N}$ denote respectively the set of all real numbers, the set of all rational numbers and the set of all natural numbers.

1. Answer the following multiple choice questions. Choose the correct option with justification. 2×10

(a) The supremum of the complement of A in $\mathbb{R}$, where $A=\{x\in\mathbb{R} : x^2-3x+2\geqslant 0\}$,

(i) is 2, (ii) is 1 (iii) is 0 (iv) does not exist.

(b) Which of the following sets is countably infinite?

(i) $\{x^2 : x\in[0,1]\}$

(ii) $\{(5+\sqrt{7})t : t\in\mathbb{Q}\}$

(iii) $(0,1]\cap(\mathbb{R}\setminus\mathbb{Q})$

(iv) $\left\{\frac{1}{x} : x\in(0,\infty)\right\}$.

(c) Let $A=\{x\in\mathbb{R} : 0<x\leqslant 1\}$ and $B=\left\{\frac{1}{n^n} : n\in\mathbb{N}\right\}$. Then $A\setminus B$ is

(i) an open set

(ii) a closed set

(iii) both open and closed

(iv) neither open nor closed.

(d) Number of limit points of the set $S=\left\{7+\frac{1}{n^2} : n\in\mathbb{N}\right\}\cup\left\{7-\frac{1}{n} : n\in\mathbb{N}\right\}$ is

(i) 1 (ii) 2 (iii) 0 (iv) infinite.

(e) Which of the following statements is true?

(i) Convergent sequences are monotonic.

(ii) Monotonic sequences are convergent.

(iii) Every sequence has a convergent subsequence.

(iv) Convergent sequences are Cauchy sequences.

(f) Lim sup of the sequence $\left\{\dfrac{1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}}{n}\right\}_{n=1}^{\infty}$ is

(i) 0 (ii) 1 (iii) ∞ (iv) $\dfrac{1}{\sqrt{2}}$.

(g) Which one of the following is true?

(i) $\left\{\dfrac{n}{2}-\left[\dfrac{n}{2}\right]\right\}_{n=1}^{\infty}$ is convergent, here $[x]$ denotes the greatest integer less than or equal to x.

(ii) $\{x_n\}_{n=1}^{\infty}$, where $x_n = \begin{cases} 5+\frac{1}{n} & \text{if } n \text{ is even} \\ \frac{1}{5}+n & \text{if } n \text{ is odd} \end{cases}$, is convergent.

(iii) $\left\{\dfrac{n}{n+1}\right\}_{n=1}^{\infty}$ is not a Cauchy sequence

(iv) $\left\{\dfrac{\sin n}{n}\right\}_{n=1}^{\infty}$ converges to 0.

(h) The set of all subsequential limits of $\left\{\cos\dfrac{n\pi}{2}\right\}_n$ is

(i) open (ii) closed (iii) both open and closed (iv) empty.

(i) The infinite series $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\ldots..$ converges to

(i) $\dfrac{1}{2}$ (ii) 1 (iii) $\dfrac{3}{2}$ (iv) $\dfrac{1}{3}$.

(j) Let $\sum\limits_{n=1}^{\infty} u_n^2$ and $\sum\limits_{n=1}^{\infty} w_n^2$ be both convergent series. Then $\sum\limits_{n=1}^{\infty} u_n w_n$ is

(i) absolutely convergent

(ii) conditionally convergent

(iii) divergent to ∞

(iv) oscillatory.

2. Answer *any four* questions :

(a) Define L.U.B. of a non-empty bounded above set of real numbers. State L.U.B. Axiom of $\mathbb{R}$. Use it, to prove that if $x, y \in \mathbb{R}$ with $x > 0$, then there exists $n \in \mathbb{N}$ such that $nx > y$. 1+1+3

(b) What do you mean by a countable set? Prove that the open interval (0, 1) is not countable. 1+4

(c) Examine whether the set $\{x \in \mathbb{R} : \cos 2x = 0\}$ is a closed set. Prove that every uncountable set of real numbers has a limit point. 2+3

(d) (i) Let A, B be two non-empty bounded subsets of $\mathbb{R}$. Prove that Sup $(A - B)$ = Sup A – inf B, where $A - B = \{x - y : x \in A, y \in B\}$.

(ii) Find the Sup A and inf A, where $A = \left\{x \in : \sin\frac{1}{x} = 0\right\}$. 3+2

(e) Let A be a subset of $\mathbb{R}$. Prove that an interior point of A is a limit point of A. Is the converse true? Justify your answer. 3+2

(f) (i) Give an example of a set of real numbers which has exactly 3 limit points and the set is closed.

(ii) Prove or disprove : $\bigcap_{n=1}^{\infty}\left(0, \frac{1}{n}\right)$ is a closed set. 3+2

(g) Prove that between any two distinct real numbers, there lie a rational number and an irrational number. 3+2

3. Answer *any four* questions :

(a) (i) Prove or disprove : If $\{x_n\}$ is convergent and $\{y_n\}$ is bounded, then $\{x_n y_n\}$ is convergent.

(ii) Find the value of $\lim_{n\to\infty} n^{1/n}$. 2+3

(b) A sequence $\{x_n\}$ is defined by $x_1 = \sqrt{7}, x_{n+1} = \sqrt{x_n + 7}$ for all $n \geqslant 1$. Show that the sequence converges to the positive root of $x^2 - x - 7 = 0$. 5

(c) (i) Prove or disprove : Every sequence of real numbers having unique limit point is convergent.

(ii) Let $\{u_n\}_{n=1}^{\infty}$ and $\{w_n\}_{n=1}^{\infty}$ be two real sequences such that $\lim_{n\to\infty}(u_n - w_n) = 0$ and $u_n \geqslant 1$, for all $n \in \mathbb{N}$. Prove that $\lim_{n\to\infty}\frac{w_n}{u_n} = 1$. 3+2

(d) (i) Prove that every Cauchy sequence of real numbers is bounded. Is the converse true? Justify your answer.

(ii) Prove or disprove : Let $\{w_n\}_{n=1}^{\infty}$ be a convergent sequence of real numbers with limit 0. Then $\left\{w_n^5\right\}_{n=1}^{\infty}$ is convergent and converges to 0. (2+1)+2

(e) (i) Give an example of a sequence of irrational numbers that converges to a rational number.

(ii) Prove or disprove : A sequence of real numbers cannot have exactly three subsequential limits. 2+3

(f) Define subsequence of a sequence of real numbers. Prove that a bounded sequence $\{a_n\}_n$ is convergent if and only if $\overline{\lim}\, a_n = \underline{\lim}\, a_n$. 1+(2+2)

(g) Define lim sup of a sequence. Find lim sup and lim inf (if exists) of the sequence

$$\{x_n\}_{n=1}^{\infty} \text{ where } x_n = \left(1-\frac{1}{n}\right)\sin\frac{n\pi}{2}, \text{ for all } n \in \mathbb{N}.$$ 1+2+2

4. Answer ***any one*** question :

(a) (i) Examine the convergence of $\sum_{n=1}^{\infty} \frac{1}{n}\sin\frac{1}{n}$.

(ii) Prove or disprove : Let $\sum_{n=1}^{\infty} u_n$ be a series of positive real numbers such that $\sum_{n=1}^{\infty} u_n^2$ is convergent. Then $\sum_{n=1}^{\infty} u_n$ is convergent. 3+2

(b) State and prove Leibnitz's test. Test the convergence of $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{1}{n}$. 4+1

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