2021

MATHEMATICS — HONOURS

Fifth Paper

(Module - IX)

Full Marks : 50

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

$\mathbb{R}$, $\mathbb{N}$ denote the set of real numbers and the set of natural numbers respectively.

Answer ***question no.*** **1** and ***any four*** questions from the rest.

1. (a) Answer ***any two*** questions :

(i) Prove or disprove : $T = \left\{1 - \frac{1}{n^2} : n \in \mathbb{N}\right\}$ is compact. 2

(ii) Correct or justify : If a real valued function f is bounded in some closed interval $[a, b]$ in $\mathbb{R}$ then f is a function of bounded variation in [a, b]. 2

(iii) Correct or justify : The power series $x + \frac{x^2}{2^2} + \frac{x^3}{3^3} + \frac{x^4}{4^4} \ldots$ is everywhere convergent. 2

(iv) Discuss the continuity of the limit function of the sequence of functions $\{f_n\}_n$ defined by $f_n(x) = \frac{x^{2n}}{1 + x^{2n}}$ on [0, 1]. 2

(b) Answer ***any two*** questions :

(i) Examine whether $\lim\limits_{x \to 0} \frac{\int_0^{x^2} e^{\sqrt{1+t}}\,dt}{x^2} = e.$ 3

(ii) If f is differentiable on [0, 1], then $\int_0^1 f'(x)\,dx = f(1) - f(0).$ 3

(iii) Cite with justification an example of a function f such that $\frac{1}{f}$ is Riemann integrable but f is not so over its domain. 3

(iv) Let $H = (0, 1) \subseteq \mathbb{R}$ and $\mathcal{G} = \{I_x : x \in H\}$ where $I_x = \left(\frac{x}{2}, \frac{x+1}{2}\right)$. Verify whether $\mathcal{G}$ is an open cover of H. 3

2. (a) If S is a bounded and closed set of real numbers, then prove that every infinite open cover of S has a finite subcover. 5

(b) Let $T = \left\{x \in \mathbb{R} : \cos\frac{1}{x} = 0\right\} \cup \{0\}$. Is $\mathbb{R} \setminus T$ compact? Justify your answer. 2

(c) Examine whether the following function is of bounded variation :

$f : [0, 1] \to \mathbb{R}$ defined by $f(x) = \begin{cases} x\sin\frac{\pi}{x}, & x \in (0, 1] \\ 0, & x = 0 \end{cases}$. 3

3. (a) If two functions f and g are Riemann integrable on $[a, b]$, use Lebesgue's theorem to prove that $|f| - fg$ is Riemann integrable on $[a, b]$. 4

(b) Let a function $f : [0, 3] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x & \text{for } 0 \le x \le 1 \\ 1 & \text{for } 1 < x \le 2 \\ x-1 & \text{for } 2 < x \le 3 \end{cases}$ and

let $F(x) = \int_0^x f(t)\, dt$ for $0 \le x \le 3$. Verify whether F is derivable on $[0, 3]$. 3

(c) Let f and g be continuous functions on a closed interval $[a, b]$ and $\int_a^b f(x)\, dx = \int_a^b g(x)\, dx$.

Show that there exist a point $c \in [a, b]$ for which $f(c) = g(c)$. 3

4. (a) State and prove Darboux's Theorem on upper Riemann integral. 1+4

(b) Give example of a Riemann integrable function that has no primitive. 2

(c) Show that $\left| \int_0^{\pi/2} \sin x \cos(x^2)\, dx \right| \le \frac{1}{2}$ 3

5. (a) Give examples (with justification) of Riemann integrable functions f, g on $[0, 1]$ such that $\int_0^1 |f - g| = 0$, but $f \ne g$. 3

(b) Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable over $[a, b]$ and $g : [a, b] \to \mathbb{R}$ be a function such that 'g' differs from 'f' at finitely many points of $[a, b]$. Prove that g is also Riemann integrable over $[a, b]$ and $\int_a^b f = \int_a^b g$. 2+2

(c) Cite with justification an example of a function f such that $|f|$ is Riemann integrable but f is not so over its domain. 3

6. (a) Examine the applicability of Weierstrass' form of Second Mean Value Theorem of Integral Calculus for $\int_0^\pi x^2 \sin x dx$. 3

(b) State Dini's Theorem on sequence of real valued functions. If $f_n(x) = x^n(1-x)$, where $\{f_n\}_n$ is a sequence of functions defined on [0, 1] then by using Dini's theorem, prove that $f_n \to 0$ uniformly on [0,1]. 1+3

(c) A sequence of functions $\{f_n\}_n$ is defined by $f_n(x) = \sqrt{x^2 + \frac{1}{n^2}}$, where $x \in [-1, 1]$. Show that $\{f_n\}_n$ is uniformly convergent on [–1, 1]. 3

7. (a) State Dirichlet's test on uniform convergence for series of functions. Prove that the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n}$ is uniformly convergent on any closed interval $[a, b]$ contained in the open interval $(0, 2\pi)$. 2+3

(b) Correct or justify :

If $\sum_{n=0}^{\infty} |a_n|$ is convergent then $\int_0^1 \left(\sum_{n=0}^{\infty} a_n x^n \right) dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1}$ 3

(c) Prove or disprove : The function defined by $f(x) = \sum_{n=1}^{\infty} \frac{\cos nx}{10^n}$, $x \in \mathbb{R}$ is continuous everywhere. 2

8. (a) Find the radius of convergence of the power series $x + \frac{(2!)^2 x^2}{4!} + \frac{(3!)^2 x^3}{6!} + \ldots + \frac{(n!)^2 x^n}{(2n)!} + \ldots$ 3

(b) Assuming the power series expansion for $(1-x^2)^{-1/2}$ as

$$\frac{1}{\sqrt{1-x^2}} = 1 + \frac{1}{2}x^2 + \frac{1.3}{2.4}x^4 + \frac{1.3.5}{2.4.6}x^6 + \ldots\ldots ;\ |x| < 1.$$

Obtain the power series for $\sin^{-1}x$ in (–1, 1). 4

(c) Correct or justify : If $\sum_{n=0}^{\infty} a_n x^n$ converges at $c \in \mathbb{R} \setminus \{0\}$, then it converges absolutely for all x such that $|x| < |c|$. 3