2020

MATHEMATICS — HONOURS

Paper : CC-12

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

1. Choose the correct answer and justify (1 mark for right answer and 1 mark for justification) : 2×10

(a) Let G be a group and $f: G \to G$ be an automorphism such that $f(x) = x^n$ where n is a fixed integer. Then

(i) G is commutative (ii) $a^n \in Z(G)$ for all $a \in G$

(iii) $a^{n-1} \in Z(G)$ for all $a \in G$ (iv) none of these.

(b) Let G be a cyclic group of order 2021. Then the number of automorphisms defined on G is

(i) 2020 (ii) 1932 (iii) 1 (iv) 1680.

(c) The number of elements of order 7 in a group of order 28 is

(i) 1 (ii) 6 (iii) 7 (iv) 27.

(d) Order of the element $\left(a, (1\,2\,3)\right) \in K_4 \times S_3$ is

(i) 2 (ii) 3 (iii) 5 (iv) 6.

(e) If G be an infinite cyclic group, then the Aut (G) is a group of order

(i) 1 (ii) 2 (iii) 3 (iv) infinite.

(f) The orthogonal component of $W = \left\{(x, y, z) \in \mathbb{R}^3 \middle/ x + y - z = 0 \text{ and } x - 2y + z = 0\right\}$ in $\mathbb{R}^3$ is

(i) $\left\{(x, y, z) \in \mathbb{R}^3 \middle/ x + y + z = 0\right\}$

(ii) $\left\{(x, y, z) \in \mathbb{R}^3 \middle/ x + 2y + 3z = 0\right\}$

(iii) $\left\{(x, y, z) \in \mathbb{R}^3 \middle/ \frac{x}{1} = \frac{y}{2} = \frac{z}{3}\right\}$

(iv) None of the above.

(g) The minimal polynomial of the matrix $\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$ is

(i) $(x+1)(x-2)$ (ii) $(x-1)(x-2)$ (iii) $(x-1)(x-2)^2$ (iv) $(x-1)(x-2)(x-3)$.

(h) Let T be a linear operator on $\mathbb{C}^3$ defined by $T(x, y, z) = (2x, 0, x)$, then the adjoint operator T^* of T is

(i) $T^*(x, y, z) = (2x+z, 0, 0)$ (ii) $T^*(x, y, z) = (2x, 0, z)$

(iii) $T^*(x, y, z) = (2x, 2y, 2z)$ (iv) $T^*(x, y, z) = (0, 0, 2x+z)$.

(i) Signature of the quadratic form $5x^2 + y^2 + 10z^2 - 4yz - 10zx$ is

(i) 1 (ii) 2 (iii) 3 (iv) 4.

(j) If $B = \{v_1, v_2, v_3\}$ is an ordered basis for $\mathbb{C}^3$ defined by $v_1 = (1, 0, -1), v_2 = (1, 1, 1), v_3 = (2, 2, 0)$, then the dual basis $\{f_1, f_2, f_3\}$ of B is given by

(i) $f_1(x, y, z) = x + y;\ f_2(x, y, z) = x + y + z;\ f_3(x, y, z) = \dfrac{x - y + z}{2}$

(ii) $f_1(x, y, z) = x - y;\ f_2(x, y, z) = x - y + z;\ f_3(x, y, z) = -\dfrac{1}{2}x + y - \dfrac{1}{2}z$

(iii) $f_1(x, y, z) = x + y;\ f_2(x, y, z) = -\dfrac{1}{2}x + y - \dfrac{1}{2}z;\ f_3(x, y, z) = x - y - z$

(iv) None of the above.

Unit - I

(Group Theory)

2. Answer ***any four*** questions :

(a) (i) Show that $|\text{Aut}(Z_n)| = \phi(n)$ where ϕ is the Euler ϕ-function.

(ii) Give examples of two groups G and H such that $G \not\simeq H$ but Aut $(G) \simeq$ Aut (H). 3+2

(b) (i) Show that $Inn\,(S_3) \simeq S_3$.

(ii) Let G be a group. If $Inn\,(G)$ is cyclic, then show that G must be abelian. 3+2

(c) Prove that $Z_m \times Z_n$ is cyclic if and only if $gcd\,(m, n) = 1$. Is $Z \times Z$ cyclic? Justify. 4+1

(d) Let G be a group, H and K be normal subgroups of G such that $G = HK$. Let $H \cap K = N$. Show that $G/N \simeq H/N \times K/N$. 5

(e) (i) Let G be a commutative group of order 99. Show that G has a unique normal subgroup H of order 11.

(ii) Show that $8\mathbb{Z}/56\mathbb{Z} \simeq \mathbb{Z}_7$. 2+3

(f) Show that for any prime p, there exist only two non-isomorphic groups of order p^2. 5

(g) (i) If G is the internal direct product of $N_1, N_2, ..., N_k$ and if $a \in N_i$, $b \in N_j$ for $i \neq j$, then prove that $N_i \cap N_j = \{e\}$ and $ab = ba$.

(ii) Let p, q be odd primes and let m and n be positive integers. Is $U(p^m) \times U(q^n)$ cyclic? Justify. Here $U(n)$ denotes the group of units modulo n. 3+2

Unit - II

(Linear Algebra)

3. Answer ***any five*** questions :

(a) Reduce the equation $7x^2 - 2xy + 7y^2 - 16x + 16y - 8 = 0$ into canonical form and determine the nature of the conic. 5

(b) (i) Prove that any two matrix representations of a bilinear form are congruent.

(ii) Let $f(x, y) = x^2 + y^2 + xy$. Find the Hessian matrix of f at (0, 0) and show that f has a local minimum at the origin. 2+3

(c) Let W be the subspace of $\mathbb{R}^3$ spanned by (1, 1, 0) and (0, 1, 1). Find a basis of the annihilator of W. 5

(d) (i) Let β be a basis for a finite-dimensional inner product space. Prove that if $\langle x, z\rangle = 0$ for all $z \in \beta$, then $x = 0$.

(ii) Show that the sum of two inner products is again an inner product. 3+2

(e) (i) Let V be a vector space over F, $\beta = \{v_1, v_2 ...., v_n\}$ be a basis of V and $\beta^* = \{f_1, f_2 ...., f_n\}$ be the dual basis. Then show that for every $v \in V, v = f_1(v)v_1 + f_2(v)v_2 + ... + f_n(v)v_n$.

(ii) Consider the linear transformation $T : M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R})$ defined by $T(A) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} A$ for all $A \in M_{2\times2}(\mathbb{R})$, check whether the subspace $W = \{A \in M_{2\times2}(\mathbb{R}) / A^t = A\}$ is T-invariant. 2+3

(f) Diagonalise the symmetric matrix $A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$. 5

(g) (i) Use Gram–Schmidt orthonormalization process to find an orthonormal basis of $\mathbb{R}^3$ from the basis $\{(1,0,1),(1,1,1),(1,3,4)\}$.

(ii) Find the adjoint of the linear operator $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$T(x, y, z) = (x + y - z, x - z, y - z)$ for all $(x, y, z) \in \mathbb{R}^3$. 3+2

(h) Show that the matrix $A = \begin{pmatrix} 3 & 1 & -2 \\ -1 & 0 & 5 \\ -1 & -1 & 4 \end{pmatrix}$ has a Jordan canonical form. Find a Jordan canonical form of A. What are the number of distinct Jordan canonical forms of A? 1+3+1