2018

MATHEMATICS – HONOURS

First Paper

(Module - I)

Full Marks : 50

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

Group - A

(Marks : 35)

Answer ***any seven*** questions.

1. (a) Prove that if the equation $qx + ry = s$ has integral solution then $gcd\,(q, r)$ divides s, where q, r, s are integers such that q and r are not both zero. 2

(b) If $a \equiv b \pmod{m}$ then show that $a^n \equiv b^n \pmod{m}$ for all positive integers n. Prove by an example that the converse of the above theorem is not true. 2+1

2. (a) State Fermat's theorem. Using this theorem prove that $n^{12} - 1$ is divisible by 7 if $gcd\,(n, 7) = 1$. 1+2

(b) Find the least positive residue in $2^{37} \pmod{19}$. 2

3. (a) Prove that $\phi(n)$ is an even integer if $n > 2$. (ϕ is the Euler's Phi function). 3

(b) What is the remainder when $6 \times 7^{32} + 7 \times 9^{45}$ is divided by 4? 2

4. Show that $n^n > 1\cdot3\cdot5.....(2n-1)$ when $n > 1$. 5

5. (a) If α and β be the roots of the equation $x^2 - 2x\cos\theta + 1 = 0$, then find the equation whose roots are α^n and β^n. 3

(b) Find the general values of Log (-1) and hence find the value of Log (-1). 1+1

6. (a) If $a\ (\neq 0)$, z_1, z_2 be complex, examine the validity of the relation $a^{z_1} \cdot a^{z_2} = a^{z_1+z_2}$. 3

(b) Show that the sum of the 99th powers of the roots of the equation $x^5 = 1$ is zero. 2

7. (a) If $u + iv = \cot(x + iy)$, prove that

$$u = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

and $$v = -\frac{\sinh 2y}{\cosh 2y - \cos 2x}.$$ 3

(b) Find $\cos^{-1} i$. 2

8. (a) If a, b, c be all positive real numbers prove that $\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq 6$. 2

(b) Find the maximum value of $(7 - x)^4 (2 + x)^5$ when x lies between (-2) and 7. 3

9. If $a_1, a_2, \ldots \ldots, a_n$ be n positive real numbers and $p_1, p_2, \ldots \ldots, p_n$ be n positive rational numbers, then prove that $\frac{p_1 a_1 + p_2 a_2 + \ldots \ldots + p_n a_n}{p_1 + p_2 + \ldots \ldots + p_n} \geq \left(a_1^{p_1} \cdot a_2^{p_2} \cdot \ldots \ldots \cdot a_n^{p_n}\right)^{1/(p_1 + p_2 + \ldots \ldots + p_n)}$. 5

10. Solve the equation $x^5 - 1 = 0$ and deduce the value of $\cos\frac{\pi}{5}$ and $\cos\frac{2\pi}{5}$. 5

11. If $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$ be the roots of $x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_{n-1} x + p_n = 0$ then find the values of $\left(\alpha_1^2 + 1\right)\left(\alpha_2^2 + 1\right) \ldots \ldots \left(\alpha_n^2 + 1\right)$. 5

12. (a) If $x^4 + ax^2 + bx + c$ has a factor of the form $(x - \alpha)^2$ show that $8a^3 + 27b^2 = 0$. 2

(b) Using Sturm's theorem find the number of real roots of the equation $x^4 + 8x^2 - 9 = 0$. 3

13. (a) Prove that if $f(x)$ be a polynomial of degree n and $f(x) = x^n f\left(\frac{1}{x}\right)$ then $f(x) = 0$ is a reciprocal equation of the first type. 2

(b) Solve the equation $x^4 - 4x^3 + 6x^2 + 4x - 7 = 0$ by Ferrari's method. 3

Group - B

(Marks : 15)

Answer ***any three*** questions.

14. (a) If A, B, Z be three non-empty sets such that $Z \cap A = Z \cap B$ and $Z \cup A = Z \cup B$ then prove that $A = B$. 3

(b) Prove that the inverse of an equivalence relation is an equivalence relation. 2

15. (a) If $f: S \rightarrow S$ be surjective where S is a finite set, then show that f is injective. 2

(b) Let $A = R - \left\{-\frac{1}{2}\right\}, B = R - \left\{\frac{1}{2}\right\}$. Let $f: A \rightarrow B$ be defined by $f(x) = \frac{x-3}{2x+1}, \forall x \in A$. Does f^{-1} exist? Justify your answer. 3

16. (a) Let G be a finite group and a, b $\in$ G. If $b = gag^{-1}$ for some $g \in G$, then prove that $o(a) = o(b)$. 3

(b) If G be a finite group with identity element e, then prove that there exists a positive integer m such that $a^m = e \, \forall \, a \in G$. 2

17. Let H and K be two subgroups of a group (G, o). Then prove that HUK is a subgroup of G if and only if $H \subseteq K$ or $K \subseteq H$. 5

18. Let $(G, \circ)$ be a group. Prove that a non-empty subset H of G forms a subgroup of $(G, \circ)$ if and only if $a \in H, b \in H \Rightarrow a \circ b^{-1} \in H$. 1+4