2018

MATHEMATICS – HONOURS

Fourth Paper

(Module-VII)

Full Marks : 50

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as applicable.

ℝ *denotes the set of real number*

Group A

(Marks – 30)

Answer **any six** questions.

1. (a) Let $S = \{x, x) \in \mathbb{R}^2 : x > 0\}$. Does the set contain any interior point? Is it closed? Justify your answer.

(b) Let $f(x,y) = \begin{cases} (ax+by)\sin\frac{x}{y}, \text{if } y \neq 0 \\ 0 \quad, \text{if } y = 0 \end{cases}$

Find $\underset{(x,y)\to(0,0)}{Lt} f(x,y)$. (1+1)+3

2. Correct or justify : If $f : S \to \mathbb{R}$ $(S \subseteq \mathbb{R}^2)$ be a function and $(a,b) \in S$ be such that the functions $f(x,b)$ and $f(a,y)$ are continuous at $x = a$ and $y = b$ respectively, then f is continuous at (a,b). 5

3. Show that the function $f(x,y) = |xy|^{3/2}, (x,y) \in \mathbb{R}^2$ is differentiable at (0, 0). 5

4. Let $f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0 \quad, & \text{if } (x,y) = (0,0) \end{cases}$

Using this function show that the conditions of Schwarz's theorem are sufficient only for the equality of the mixed partial derivatives. 5

5. Let $u = \sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$. Show that $\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)^2 u = -\frac{\sin u \cdot \cos 2u}{4\cos^3 u}$. 5

6. If $x = r\cos\theta, y = r\sin\theta$ and u be a real-valued function in x and y, then show that

$$\frac{\partial^2 u}{dx^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial u}{\partial r}+\frac{1}{r^2}\cdot\frac{\partial^2 u}{\partial \theta^2}$$ 5

7. Prove that the following three functions u, v, w are functionally related.

$u=\frac{x}{y-z}, v=\frac{y}{z-x}, w=\frac{z}{x-y}$. Also find the relation among u, v, w. 2+3

8. If $u=\frac{x}{\sqrt{1-r^2}}$, $v=\frac{y}{\sqrt{1-r^2}}$, $w=\frac{z}{\sqrt{1-r^2}}$, where $r^2=x^2+y^2+z^2$ then show that $\frac{\partial(u,v,w)}{\partial(x,y,z)}=\frac{1}{(1-r^2)^{5/2}}$. 5

9. (a) Examine the existence of implicit function near the point $\left(\frac{\pi}{3},2\right)$ for the equation

$$y^3\text{Cos}x + y^2\text{Sin}^2x - 7 = 0.$$

Find $\frac{dy}{dx}$ at $\left(\frac{\pi}{3},2\right)$, if exists.

(b) Let $u(x,y)=\begin{cases}\frac{(x+y)^2(x-y)}{x^2+y^2}, & \text{if } x^2+y^2\neq 0\\ 0, & \text{if } x^2+y^2=0\end{cases}$

Prove that $u(x,y)=0$ does not define y as a single-valued function of x near the origin. (2+1)+2

10. State and prove Taylor's theorem for a real-valued function of two real variables. 1+4

11. Use the method of Lagrange's multipliers to show that the lengths of the semiaxes of the ellipse $ax^2+2bxy+cy^2=1$ are the square roots of the roots of the equation

$$\begin{vmatrix} a\lambda-1 & b\lambda \\ b\lambda & c\lambda-1\end{vmatrix}=0.$$ 5

Group B

(Marks – 20)

Answer **any four** questions.

12. If ρ_1 and ρ_2 be the radii of curvature at the ends P and Q of conjugate semi-diameters CP and CQ respectively of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then prove that $\rho_1^{2/3}+\rho_2^{2/3}=\frac{a^2+b^2}{(ab)^{2/3}}$, where C is the centre of the ellipse. 5

13. Prove that the curve defined by

$$y=\sqrt{1+x^2}\sin\frac{1}{x}, x\neq 0$$
$$=0, \qquad x=0$$

has no asymptote parallel to the y-axis and its only asymptotes are $y=\pm 1$. 5

14. Prove that the pedal equation of a circle with respect to a point on the circumference is $pd=r^2$, where d is the diameter of the circle. 5

15. Show that the envelope of straight lines which join the extremities of a pair of conjugate diameters of an ellipse is a similar ellipse. 5

16. Find the area common to the circles $r=a\sqrt{2}$ and $r=2a\cos\theta$. 5

17. Find the range of values of x for which the curve $y=x^4-16x^3+42x^2+12x+1$ is concave upwards or downwards. Find also the points of inflexion, if any. 5

18. Find the centre of gravity of an arc of the astroid $x=a\cos^3 t, y=a\sin^3 t$ lying in the first quadrant. 5