2018

MATHEMATICS – HONOURS

Second Paper

(Module - III)

Full Marks : 50

The figures in the margin indicate full marks.
Candidates are required to give their answers in their own words as far as practicable.

$\mathbb{R}$, $\mathbb{Q}$, $\mathbb{N}$ denote the sets of real numbers, rational numbers and natural numbers respectively.

Group – A

(Marks : 40)

Answer ***any four*** questions.

1. (a) State LUB Axiom of the set of real numbers. Show that $\mathbb{R} - \mathbb{Q}$, the set of all irrationals, is unbounded. What can be said regarding boundedness of the complement of $\mathbb{R} - \mathbb{Q}$? 1+2+1

   (b) Find *sup A* and *inf A* where $A = \left\{x \in \mathbb{R} \,/\, x^2 - x - 12 < 0\right\}$ 3

   (c) Define neighbourhood of a point in $\mathbb{R}$. Check whether the set $\left\{\frac{1}{n} \middle| n \in \mathbb{N}\right\} \cup \{0\}$ is a neighbourhood of 0 or not. 1+2

2. (a) Show that every infinite set has a countably infinite subset. 3

   (b) Prove or disprove : A countable set cannot have uncountable number of limit points. 3

   (c) Prove or disprove : There is no bounded infinite set whose points are isolated. 2

   (d) Show that the set $\{x \in \mathbb{R} : \sin x = 0\}$ is countable. 2

3. (a) Let A be a countable set of real numbers. Examine whether $A \cup \left\{\bigcap_{n=1}^{\infty}\left[-\frac{1}{n+1}, \frac{1}{n+1}\right]\right\}$ is countable. 3

   (b) Prove that the complement of an open set is closed. Hence show that for any $x > 0$, the set $\{nx : n \in \mathbb{N}\}$ is a closed set. 3+2

   (c) Show that every nonempty open set can be expressed as a union of open intervals. 2

4. (a) Define Cauchy sequence. Prove that every Cauchy sequence is bounded. Is the converse true? Justify. 1+3+2

(b) Let $\{I_n\}_{n=1}^{\infty}$ be a sequence of nested closed and bounded intervals. Prove that $\bigcap_{n=1}^{\infty} I_n \neq \phi$. Is the result true for unbounded intervals? Justify. 3+1

5. (a) Prove or disprove : If $\{x_n\}$ is a bounded sequence and $\{y_n\}$ is a convergent sequence then $\{x_n y_n\}$ is a convergent sequence. 2

(b) Prove that the sequence $\{x_n\}$ defined by $x_1 = \sqrt{7}$ and $x_{n+1} = \sqrt{7 + x_n}$ for all $n \geq 1$, is convergent and converges to the positive root of the equation $x^2 - x - 7 = 0$. 4

(c) Show that a bounded sequence $\{x_n\}$ is convergent if and only if $\lim sup\, x_n = \lim inf\, x_n$. 4

6. (a) Let $f : D \to \mathbb{R}$ $(D \subseteq \mathbb{R})$ be a function such that $\lim_{n\to\infty} f(x_n) = f(a)$, for any sequence $\{x_n\}$ in D converging to $a \in D$. Show that f is continuous at a. 3

(b) Let $f : [0, 1] \to \mathbb{N}$ be a continuous function. Show that f is a constant function. 3

(c) Prove or disprove : If $f : \mathbb{R} \to \mathbb{R}$ is a function such that f^2 is continuous then f is so. 2

(d) Let f, g be two real valued continuous functions of real variable and $f(x) = g(x)\ \forall x \in \mathbb{Q}$. Show that $f(x) = g(x)\ \forall x \in \mathbb{R}$. 2

7. (a) Define Lipschitz function. Show that every Lipschitz function is uniformly continuous. 1+2

(b) Let $f(x) = |x - 2018|,\ \forall x \in \mathbb{R}$. Show that f is uniformly continuous on $\mathbb{R}$. 2

(c) Let $\{x_n\}$ be a sequence in (a, b) and $f : (a, b) \to \mathbb{R}$ be an uniformly continuous function. Prove that there exists a convergent subsequence of $\{f(x_n)\}$. 3

(d) Prove or disprove : the function $f(x) = x \sin\frac{1}{x}, x \neq 0$, is uniformly continuous on $\left(0, \frac{1}{\pi}\right)$. 2

Group – B

(Marks : 10)

8. Answer ***any two*** questions : 5×2

(a) Find the reduction formula for $\int \sec^n x\, dx$, n being a positive integer greater than 1 and hence find $\int \sec^4 x\, dx$. 3+2

(b) Evaluate : $\int \frac{dx}{5+4\sin x}$ 5

(c) Evaluate : $\int_0^a \frac{dx}{\left(x^2+a^2\right)^2}$ $(a>0)$ 5

(d) Evaluate : $\underset{n\to\infty}{\mathrm{Lt}}\left[\frac{1}{n}+\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\ldots+\frac{1}{8n}\right]$ 5
